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NUMERICAL AND EXACT SOLUTION OF BUCKLING LOAD FOR BEAM ON ELASTIC FOUNDATION

NUMERICKÉ A ANALYTICKÉ RIEŠENIE STRATY STABILITY NOSNÍKA NA PRUŽNOM PODKLADE

Abstract

In this paper we will be presented the exact solution of buckling load for supported beam on elastic foundation. Exact solution will be compared with numerical solution by FEM in our code in Matlab. Implementation of buckling to FEM will be presented here.

Abstrakt

V článku je prezentovaná analytické riešenie kritickej sily pri riešení stratu stability nosníka na pružnom podklade. Analytické riešenie je porovnané s numerickým riešením založeným na metóde konečných prvkov vo vlastnom Matlabovskom programe. Implementácia straty stability je prezentovaná v tomto článku.

Keywords

Beam, Elastic foundation, Buckling, Finite element method, Matlab, ANSYS.

1 INTRODUCTION

The research area of buckling on non-uniform columns and beams has been one important topics of extensive studies based on the reality that is closely related to the field of structural, mechanical and aeronautical engineering. Determination of practical load carrying capacity of structural members requires a detailed stability analysis in theoretical and computational manner.

2 THEORETICAL BACKGROUND

Considering a uniform homogenous beam of flexural rigidity EI, length L, and continuously restrained along its length. The restrain consists of lateral spring of stiffness k per unit length, e.g. Winkler foundation. The beam is loaded at top end by a concentrated compressive force P as shown in the Fig. 1. The governing buckling equation is given by

$$\frac{\mathrm{d}^4 w}{\mathrm{d}\,x^4}+\alpha\frac{\mathrm{d}^2 w}{\mathrm{d}\,x^2}+\beta w=0, \tag{1}$$

where: $x=\bar{x}/\mathrm{L}$, $w=w/\mathrm{L}$, $\alpha=\mathrm{PL}^2/(\mathrm{EI})$ and $\beta=\mathrm{kL}^4/(\mathrm{EI})$.

The general solution to (1) is

$$w(x)=C_1\cos(Sx)+C_2\sin(Sx)+C_3\cos(Tx)+C_4\sin(Tx), \tag{2}$$

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where

$$S=\sqrt{\frac{\alpha}{2}-\sqrt{\left(\frac{\alpha}{2}\right)^{2}-\beta}},\ T=\sqrt{\frac{\alpha}{2}+\sqrt{\left(\frac{\alpha}{2}\right)^{2}-\beta}}. \tag{3}$$

The classical boundary conditions for pinned end are as follows: $w = 0$, $\mathrm{d}^2w/\mathrm{d}x^2 = 0$. The general solution of governing equation and stability criterion is following

$$\sin T = 0. \tag{4}$$

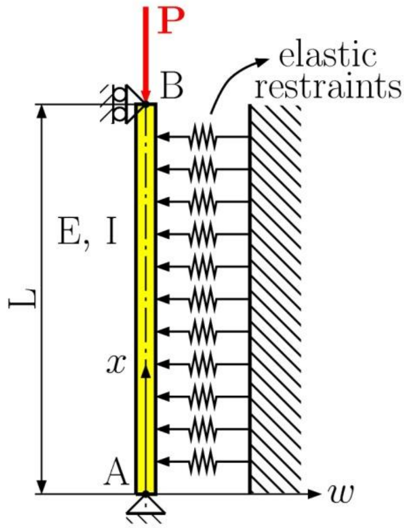

Fig. 1 Beam with continuous elastic restrains.

From Eqn. (4) the critical buckling load is following

$$\mathrm{P_{crit}} = \frac{\mathrm{kL^4} + \pi^4 \mathrm{EI}}{\pi^2 \mathrm{L}^2}. \tag{5}$$

3 FEM FORMULATION OF THE BEAM BUCKLING

The column is discretized using two-noded Euler beam elements of length L with two degrees of freedom namely transverse displacement and rotation at each node as shown in Fig 2. Let I be the moment of inertia of the beam cross sectional area. To describe the displacement at intermediate nodal points Hermite polynomial shape functions are used

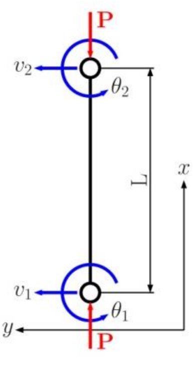

Fig. 2 Beam element.

$$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3},\quad N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2},\quad N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3},\quad N_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}. \tag{6}$$

The transverse displacement w(x) can be written as

$$\mathrm{v}(x) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{7}$$

Elementary stiffness matrix for beam is given by [3]

$$\mathbf{K}_{\mathbf{B}}^{\mathbf{e}} = \int_0^L \mathbf{B}^{\mathrm{T}} \mathrm{EI}\, \mathbf{B}\, \mathrm{d}x = \frac{\mathrm{EI}}{\mathrm{L}^3} \begin{bmatrix} 12 & 6\mathrm{L} & -12 & 6\mathrm{L} \\ 6\mathrm{L} & 4\mathrm{L}^2 & -6\mathrm{L} & 2\mathrm{L}^2 \\ -12 & -6\mathrm{L} & 12 & -6\mathrm{L} \\ 6\mathrm{L} & 2\mathrm{L}^2 & -6\mathrm{L} & 4\mathrm{L}^2 \end{bmatrix} \tag{8}$$

The elemental mass matrix for the beam [2]

$$\mathbf{M}^{\mathbf{e}} = \int_0^L \mathbf{N}^{\mathrm{T}} \rho \mathrm{A}\, \mathbf{N}\, \mathrm{d}x = \frac{\rho \mathrm{AL}}{420} \begin{bmatrix} 156 & 22\mathrm{L} & 54 & -13\mathrm{L} \\ 22\mathrm{L} & 4\mathrm{L}^2 & 13\mathrm{L} & -3\mathrm{L}^2 \\ 54 & 13\mathrm{L} & 156 & -22\mathrm{L} \\ -13\mathrm{L} & -3\mathrm{L}^2 & -22\mathrm{L} & 4\mathrm{L}^2 \end{bmatrix} \tag{9}$$

2.1 Elemental Geometric Stiffness Matrix

The beam is subjected to an external axial periodic force P(t) ,the elemental work done by the external periodic force P(t) is given by

$$\mathbf{w}^{\mathbf{e}} = \frac{1}{2} \int_0^L \mathrm{P(t)} \frac{\partial^2 \mathrm{w}}{\partial x^2}\, \mathrm{d}x \tag{10}$$

Substituting eq. (7) in eq. (10) and applying the Galerkin's method yields the geometric stiffness matrix is following [2]

$$\mathbf{K}_{\mathbf{g}}^{\mathbf{e}} = \int_0^L \left[\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}\right]^{\mathrm{T}} \mathrm{P} \left[\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}\right] \mathrm{d}x = \frac{\mathrm{P}}{30\mathrm{L}} \begin{bmatrix} 36 & 3\mathrm{L} & -36 & 3\mathrm{L} \\ 3\mathrm{L} & 4\mathrm{L}^2 & -3\mathrm{L} & -\mathrm{L}^2 \\ -36 & -3\mathrm{L} & 36 & -3\mathrm{L} \\ 3\mathrm{L} & -\mathrm{L}^2 & -3\mathrm{L} & 4\mathrm{L}^2 \end{bmatrix} \tag{11}$$

Where $\mathbf{K}_{\mathbf{g}}^{\mathbf{e}}$ is called geometric stiffness matrix or stability matrix or initial stress matrix.

2.2 Elemental Stiffness Matrix of Elastic Foundation

The Winkler's foundation model is easy to formulate using energy concepts. The analysis of bending of beams on an elastic foundation (Winkler's model) is developed on the assumption that:

- The strains are small.
- The resisting pressure $\mathrm{p}_{\mathrm{R}} = K\,\mathrm{v}$ in the foundation are proportional at every point to the deflection v = v(x) normal to its surface at that point, where K is the modulus of the foundation.

An area dA of the foundation surface acts like a linear spring of stiffness k /Nm^{-2}/. Hence, $k = \mathrm{p_R}\, dA/\mathrm{v} = K\,\mathrm{v}\, dA/\mathrm{v} = K\, dA$. Strain energy $\mathrm{U_R}$ /N/ in a linear spring is $\mathrm{U_R} = k\,\mathrm{v}^2/2$.

Now considering a structural element, perhaps a plate bending element or one face of a 3D solid element, which has an area A in a contact with the foundation. Lateral deflection of area A normal to the foundation, is $\mathrm{v} = \mathbf{N_f d_f}$, where $\mathbf{d_f}$ contains D.O.F. of element nodes in contact with foundation. Strain energy U /N/. in foundation over area is

$$U = \frac{1}{2}\int K\mathrm{v}^2 dA = \frac{1}{2}\mathbf{d_f^T K_f d_f} \tag{12}$$

in which the Winkler's foundation stiffness matrix for the element is

$$\mathbf{K_f} = \int K\mathbf{N_f}^T\mathbf{N_f}\, dA = \begin{bmatrix} \frac{13\mathrm{bL}K}{35} & \frac{11\mathrm{bL}^2K}{210} & \frac{9\mathrm{bL}K}{70} & -\frac{13\mathrm{bL}^2K}{420} \\ \frac{11\mathrm{bL}^2K}{210} & \frac{\mathrm{bL}^3K}{105} & \frac{13\mathrm{bL}^2K}{420} & -\frac{\mathrm{bL}^3K}{140} \\ \frac{9\mathrm{bL}K}{70} & \frac{13\mathrm{bL}^2K}{420} & \frac{13\mathrm{bL}K}{35} & -\frac{11\mathrm{bL}^2K}{210} \\ -\frac{13\mathrm{bL}^2K}{420} & -\frac{\mathrm{bL}^3K}{140} & -\frac{11\mathrm{bL}^2K}{210} & \frac{\mathrm{bL}^3K}{105} \end{bmatrix}, \tag{13}$$

where $dA = \mathrm{b}\, dx$, b is the width of the beam face contact with the foundation and L is length of the beam.

2.3 Natural Frequencies and Mode Shapes

The buckling analysis of beams considers the solution of the eigenproblem

$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = 0, \tag{14}$$

where **K** is the assembled stiffness matrix of the beam and **M** is the assembled mass matrix of the beam, we get Eigenvalues / natural frequencies and Eigenvectors / mode shapes.

2.4 Buckling load

On solving the Eigen value equation (14)

$$\left|\mathbf{K_B} + \mathbf{K_f} - \lambda\mathbf{K_G}\right|\mathbf{X} = 0, \tag{15}$$

where λ are the critical loads and **X** the buckling modes.

4 NUMERICAL SOLUTION OF THE BEAM BUCKLING ON ELASTIC FOUNDATION

Consider the beam of length L = 1 m, area A = 100 mm^2, width b = 5 mm, moment of inertia I = 833,333 mm^4, made by steel of Young’s modulus E = 200 GPa and Poisson’s ratio 0.3. Buckling problem for this paper for implemented to our Matlab Finite Element code. The finite element model is shown in the Fig. 3. In this model we change the number of beam and spring element by the parameter NDIV (number of division). The result of theoretical solution for Eqn. (4) and numerical solution is shown in the Tab. 1.

Tab. 1 Numerical and theoretical results of force P (N)

P (N) for NDIV	k				
	1e2	1e3	1e4	1e5	1e6
1	2010,00	2100,00	3000,00	12000,00	33809,50
10	1655,09	1746,28	2658,17	9114,15	26077,60

50	1655,07	1746,25	2658,15	9112,77	26032,30
Theoretical	1655,07	1746,25	2658,15	11777,10	102966,00

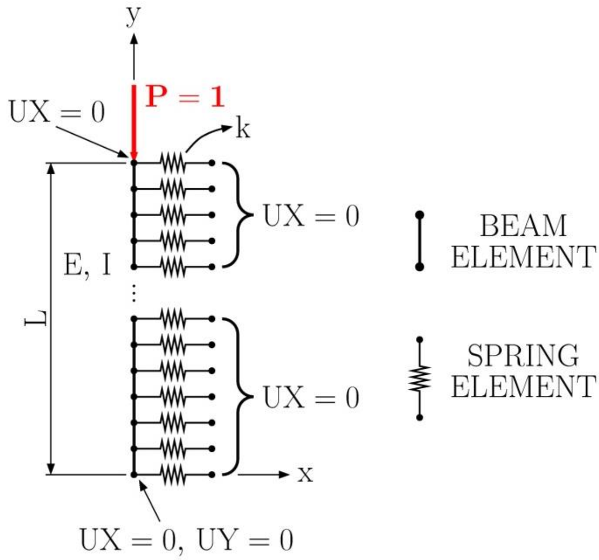

Fig. 3 FEM model implemented to MATLAB.

3 CONCLUSIONS

This paper consists of theoretical solution of pinned-pinned beam and basic matrix used for implementation for solution by finite element methods. Last part of this paper consist of compared of numerical result by own FEM program in Matlab with theoretical solution.

In numerical solution, that buckled mode shape is dependent on the spring constant k. For small values of k see Tab. 1 are the critical force equal in numerical and theoretical solutions. The accuracy of the solution in this case depends on the number of beam elements, see Fig. 4 for k = 100.

The number of halfwaves for the critical buckling mode increases with increasing values of k as it uses lesser energy than a buckled mode shape characterized by a single half wave. For k biggest of value 2e4 for beam of parameters in this paper we have different. The reason for the different results is the first mode shape is not a single half wave. The results for buckling beam on elastic foundation are correct when the first mode shape is not a single half wave.

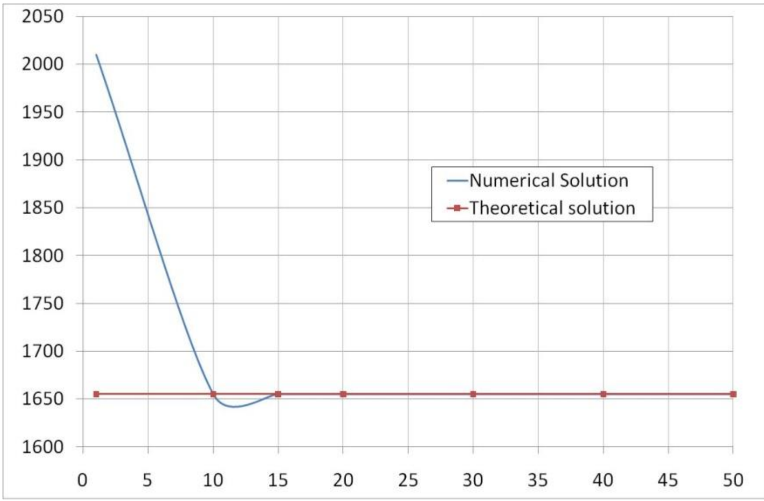

Fig. 4 FEM and theoretical solution for k = 100.

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