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# DYNAMIC ANALYSIS OF VIBRO-IMPACT GEAR DRIVE SYSTEM

# DYNAMICKÁ ANALÝZA RÁZOVÉHO KMITÁNÍ SOUSTAVY S OZUBENÝM PŘEVODEM

**Abstract**

The mechanical systems with impact motion introduce a large class of mechanical devices. Impact oscillations are usually perceived as undesirable and harmful dynamic phenomena. However, the impact motions can have positive effects (impact forming machines, drop hammers, impact presses, etc.). The contribution presents a methodology of modelling of mechanical systems with contacts which include normal forces. Such systems are described by non-smooth mathematical models, and specific numerical strategies have to be employed to solve them (smoothing method, switch method, event driven method). The methodology will be tested on a simplified model of test single-stage transmission.

**Abstrakt**

Široká třída aplikací je založena na přítomnosti rázových dějů v systému. Kmitání soustav s rázy je obvykle chápáno jako nežádoucí dynamický stav, přesto však může mít i pozitivní vliv (viz vibrolisy, buchary, atd.). Tento příspěvek představuje způsoby modelování mechanických soustav s kontakty, které uvažují normálové síly. Podobné systémy jsou popsány nehladkými matematickými modely, které vyžadují užití speciálních způsobů numerické integrace pohybových rovnic (metoda zhladčení nehladkých funkcí, přepínací model, metoda řízení integrace událostmi). Zmíněná metodologie je testována na zjednodušeném modelu testovací převodovky.

**Keywords**

Impact, mechanics, dynamics, vibration, modelling, bifurcation, gear drive

## 1 INTRODUCTION

Impact motions appear in many phenomena both in the nature and in the technical applications. In engineering mechanics, there is often necessary to struggle with dynamical effects caused by presence of construction clearances, etc. Such motions cause an increase of mechanical stress and a reduction of lifetime, and that is why they are usually perceived as undesirable phenomena. However, except for these cases, there are a lot of devices directly based on impact vibrations, e.g. impact-forming machines, drop hammers, impact presses etc. As shown on mentioned examples, it is necessary to look for suitable mathematical models of these systems.

Impact motion phenomena are a typical example of large class of technical applications called non-smooth systems. These devices are described by mathematical models which include non-smooth vector fields. Except for impacts, there are other phenomena such as dry friction that appear in dynamics of non-smooth systems. In electronics, there is an analogy with non-smooth mechanical systems - typical electrical element with non-smooth characteristics is e.g. Zener diode.

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When using standard ODE solvers to find the numerical solution of non-smooth mathematical models, there is a lot of potential problems, near discontinuity surface in particular. To find the solution of non-smooth systems, special numerical strategies are required. One of the possible ways is a formulation of mathematical model as a system of ordinary differential equation and subsequent reformulation as differential inclusion. This procedure is known as Filippov convex method. There are a few methods how to solve such systems.

## 2 MATHEMATICAL MODEL OF MECHANICAL SYSTEM WITH CLEARANCES — GENERAL DESCRIPTION

A mathematical model of vibrating discrete mechanical system with $n$ degrees of freedom is considered. In the system, there are clearances which cause non-smooth motion. We focus on special type of systems with clearance. For the motion within clearance, the mathematical model in standard matrix form is considered

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{B}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}\boldsymbol{q}(t) = \boldsymbol{f}(t), \tag{1}$$

where $t \geq 0$ is time, $\boldsymbol{M}, \boldsymbol{B}, \boldsymbol{K} \in \mathbb{R}^{n,n}$ are mass matrix, damping matrix and stiffness matrix, respectively, and $\boldsymbol{q}(t) \in \mathbb{R}^n$ is vector of generalized coordinates. Vector $\boldsymbol{f}(t)$ describes excitation forces of the system. For motion in contact phase (not in clearance), a new viscoelastic coupling activates and the damping matrix and stiffness matrix change structure to another form $\boldsymbol{B}_k, \boldsymbol{K}_k$ which describes moreover the contact-stiffness properties. It is advantageous to reformulate mathematical model of the system in another way and transform it from configuration space to state space. Generally, this is the way to rewrite system of $n$ differential equations of second order to system of $2n$ differential equations of first order. Using trivial identity $\boldsymbol{M}\dot{\boldsymbol{q}} - \boldsymbol{M}\dot{\boldsymbol{q}} = \boldsymbol{0}$, the model (1) can be rewritten into new form

$$\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{F}(t, \boldsymbol{q}), \tag{2}$$

where $\boldsymbol{u}$ is state vector, $\boldsymbol{A}$ is so-called system matrix and $\boldsymbol{F}$ is vector of right-hand side, all defined by

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}, \quad \boldsymbol{A} = -\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{E} \\ \boldsymbol{M}^{-1}\boldsymbol{K} & \boldsymbol{M}^{-1}\boldsymbol{B} \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{f} \end{bmatrix}, \tag{3}$$

where $\boldsymbol{E}$ is identity matrix. In the system with clearances, we can define switching boundary functions $h_i(\boldsymbol{u}), i = 1 \ldots n_h$ as the functions where one smooth vector field changes to another smooth vector field. These functions make the whole state space non-smooth. Further, system with one two-sided impact is considered, i.e. $n_h = 2$. In case of longitudinally vibrating system with possible contact between $i$-th and $j$-th mass, the switching boundary functions can be defined by

$$\begin{aligned} h_1(\boldsymbol{u}) &= q_j - q_i - \delta, \\ h_2(\boldsymbol{u}) &= q_i - q_j, \end{aligned} \tag{4}$$

where $\delta$ is clearance between $i$-th and $j$-th mass in initial position. Now, it is possible to formulate subspaces $\mathcal{V}_-, \mathcal{V}_+$ of state space as the spaces which are separated by the switching boundary functions $h_i(\boldsymbol{u})$ and $h_j(\boldsymbol{u})$. Both of these subspaces are smooth, but there is non-smoothness on the switching boundary. It implies that [1]

$$\begin{aligned} \mathcal{V}_- &= \{\boldsymbol{u} \in \mathbb{R}^{2n} : h_1(\boldsymbol{u}) < 0 \wedge h_2(\boldsymbol{u}) < 0\}, \\ \mathcal{V}_+ &= \{\boldsymbol{u} \in \mathbb{R}^{2n} : h_1(\boldsymbol{u}) > 0 \vee h_2(\boldsymbol{u}) > 0\}, \end{aligned} \tag{5}$$

The model in the state space can be written in following form

$$\dot{\boldsymbol{u}} = \boldsymbol{f}(t, \boldsymbol{u}) = \begin{cases} \boldsymbol{f}_-(t, \boldsymbol{u}), & \boldsymbol{u} \in \mathcal{V}_-, \\ \boldsymbol{f}_+(t, \boldsymbol{u}), & \boldsymbol{u} \in \mathcal{V}_+, \end{cases} \tag{6}$$

where $\boldsymbol{f}_{-}(t, \boldsymbol{u}), \boldsymbol{f}_{+}(t, \boldsymbol{u})$ using the coefficient matrices defined above, are vectors in the form

$$\boldsymbol{f}_{-}(t, \boldsymbol{u}) = -\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{E} \\ \boldsymbol{M}^{-1}\boldsymbol{K} & \boldsymbol{M}^{-1}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} + \boldsymbol{F}_{-} = \boldsymbol{A}_{-}\boldsymbol{u} + \boldsymbol{F}_{-}, \tag{7}$$

$$\boldsymbol{f}_{+}(t, \boldsymbol{u}) = -\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{E} \\ \boldsymbol{M}^{-1}\boldsymbol{K}_k & \boldsymbol{M}^{-1}\boldsymbol{B}_k \end{bmatrix} \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} + \boldsymbol{F}_{+} = \boldsymbol{A}_{+}\boldsymbol{u} + \boldsymbol{F}_{+}. \tag{8}$$

These matrices characterize different topology of the structure in a phase of contact and in a phase without contact - in motion with contact between bodies, there are some extra viscoelastic couplings activated. To find a solution of the system (6), the numerical methods must be used in general.

## 3 APPROACHES TO NUMERICAL SOLUTION OF NON-SMOOTH SYSTEMS

To find numerical solution of non-smooth systems, it is possible to use different approaches how to fit standard numerical solvers of ODEs for non-smooth system. In non-smooth phenomena, there is the biggest problem with numerical solution around a discontinuity surface, where the solution is susceptible to oscillate between two smooth surfaces around and to collapse.

In neighborhood of the discontinuity surface, there is often the jump in state space defined by functions like signum or Heaviside function. The first method how to find numerical solution of non-smooth system is called *smoothing method* and it is based on replacing these functions with smooth function. E.g. the sign and Heaviside function H(.) can be approximated by

$$sign(x) = \frac{2}{\pi} arctan(\varepsilon x), \quad H(x) = \frac{1}{2}(1 + sign(x)), \quad \varepsilon \gg 1. \tag{9}$$

The smoothing method is usable for large class of systems, but there are many examples, when the method does not work fine, in particular in systems with sliding surfaces, e.g. systems with dry friction.

Another approach to reach numerical solution is called switch model [1]. This method is based on direct switching between subspaces around the switching boundary. In neighborhood of the switching boundary the belt of width $2\eta$ ($\eta$ is scalar parameter of numerical method) is considered and if the boundary of the belt is crossed, the vector field is changed. In numerical integration function, there is used notation "if ... then", which defines the space, where the next step of integration is applied. There are sophisticated methods to cope with sliding modes around the switching boundary using the switch model.

Another method to find numerical solution is called event driven integration method. In this case, the numerical solution is looked for in one smooth vector field, while the event occurs. Then the integration is stopped and the kinematic quantities (displacement and velocities of all bodies) are saved. In this phase, the constitutive equations are used and generally, it is necessary to solve linear complementarity problem to find out, how the system will be moving in the future. The kinematic quantities are used as initial conditions in next integration step. The system is integrated in smooth vector field while new event occurs.

## 4 APPLICATION TO TEST SINGLE-STAGE TRANSMISSION

As an example, the above theory was applied to a mathematical model of test single-stage transmission mentioned in [2]. For only qualitative analysis of the impact motion, whole system was simplified to system with four degrees of freedom. Four discs are considered, connected by flexible shafts and gearing between stages. In gearing, two-sided impact motion is considered with kinematic transmission error in gearing $\Delta(t)$, which is considered as a harmonic function - in general, it could be considered as periodical (polyharmonic) function. Generalized coordinates are rotations $\varphi_i = q_i$ of all considered discs $I_i, i = 1, \ldots, 4$ (see fig. 1). In initial position, there is clearance $\mu$ between teeth in gearing. Deformation

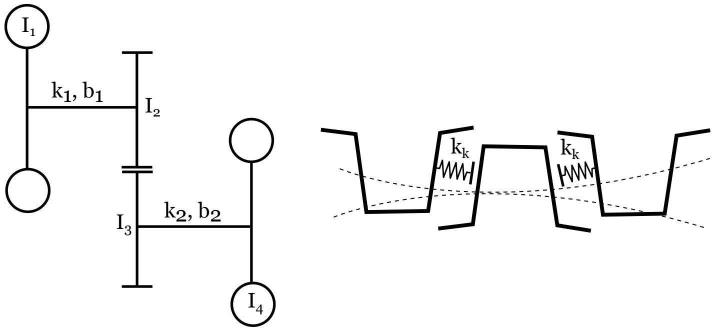


**Fig. 1** Schematic model of simplified test single-stage transmission

of gearing on mesh line is given in form

$$d_z = r_3\varphi_3 - r_2\varphi_2 + \Delta(t). \tag{10}$$

Mathematical model of system can be completed in form (1) e.g. using Lagrange's equations [3]

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{B}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{f}_I(t) + \boldsymbol{f}_E(t) + \boldsymbol{f}_N(\boldsymbol{q}), \tag{11}$$

where $\boldsymbol{M}, \boldsymbol{B}, \boldsymbol{K}$ are mass matrix, damping matrix and stiffness matrix in form

$$\boldsymbol{M} = diag\{I_i\}, \quad ,i = 1,\dots 4, \quad \boldsymbol{K} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix}, \quad \boldsymbol{B} = \alpha\boldsymbol{K}, \quad \alpha \in \mathbb{R}^+. \tag{12}$$

In (12), $I_i, i = 1,\dots 4$, are moments of inertia of discs, $k_1, k_2$ are stiffnesses of shafts, $k_k$ is contact stiffness of gearing and $\alpha$ is coefficient of proportional damping. On right-hand side of (11) are vector of internal excitation by kinematic transmission error of gearing $\boldsymbol{f}_I(t)$, vector of external excitation forces $\boldsymbol{f}_E(t)$ and vector of nonlinear forces $\boldsymbol{f}_N(\boldsymbol{q})$, all defined by

$$\boldsymbol{f}_I(t) = -k_z\Delta(t)\begin{bmatrix} 0 \\ -r_p \\ r_k \\ 0 \end{bmatrix}, \quad \boldsymbol{f}_E(t) \in \mathbb{R}^4, \quad \boldsymbol{f}_N(t) = \begin{bmatrix} 0 \\ r_p f_n \\ -r_k f_n \\ 0 \end{bmatrix}, \tag{13}$$

where $r_p = r_2, r_k = r_3$ are radii of pinion and gear. Non-linear function of force transmitted by gearing $f_n$ is given as

$$f_n = \begin{cases} k_z d_z, & d_z > 0, \\ 0, & d_z \in \langle -\mu, 0\rangle, \\ k_z(d_z - \mu), & d_z < -\mu. \end{cases} \tag{14}$$

Switching boundary functions are defined analogically to (4) in the form fit to torsional coordinates as $h_1 = d_z, h_2 = d_z + \mu$. To use smoothing method, it is necessary to reformulate problem as one analytical function. The non-smooth function $f_n$ can be rewritten using Heaviside function $H(.)$ as

$$f_n = k_z d_z - k_z d_z H(-d_z) + k_z(d_z + \mu)H(-(d_z + \mu)). \tag{15}$$

For switch method, the notation of $f_n$ directly from (14) was used for numerical implementation.

## 4.1 Discussion of gained results

The numerical solution was accomplished with a relative tolerance of $10^{-4}$ and absolute tolerance of $10^{-4}$, using variable-step Runge-Kutta method of fourth order implemented in MATLAB as ode45. Simulation of motion was accomplished for 0.2 s. According numerical results, the time domain $\langle 0, 0.2 \rangle$ s is large enough to die down of transient motions. In case of smoothing method, two possible notations were used for Heaviside function. First, the approximation (15) with $\varepsilon = 10^{11}$ was used, second the standard Heaviside function implemented in MATLAB was used. The computational times and numbers of integration steps are compared in tab. 1 for the mentioned approaches.

**Tab. 1** Efficiency of used numerical approaches

| Method | Computational time [s] | Number of integration steps |
|---|---|---|
| Switch method | 7.675758 | 25489 |
| Smoothing method | 6.016781 | 25509 |
| Switch method (using Heaviside f.) | 7.691480 | 25489 |

The numerical simulation of nonlinear behavior of the model was guided by two parameters, by external constant torque $M$ and by rotational speed of the driving shaft $n$. The external torque $M$ acts on the flywheels and corresponds to a given power transmitted by the gear drive. These two parameters were used to perform a qualitative analysis of the gear drive response to internal transmission error in gearing and mutually to external constant torque. As an indicator of system nonlinear behavior, the bifurcation diagram is used along with the phase trajectories.

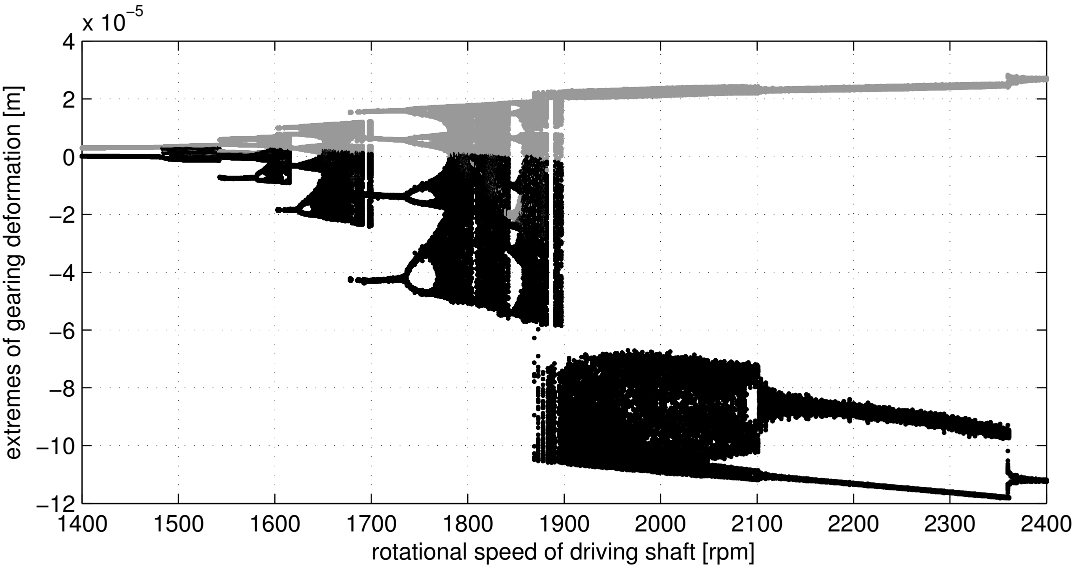


**Fig. 2** Bifurcation diagram - maxima (gray) and minima (black) of deformation $d_z$ in gearing in dependence on revolutions per minute

Fig. 2 displays a bifurcation diagram which was constructed for chosen value of the external torque $M$ =100 Nm. It shows maxima and minima of deformation $d_z$ in gearing in dependence on revolutions per minute. From this diagram, changes in quality of oscillations are obvious. There appear following phenomena. There are jumps in the amplitude of the response which are followed by period-doubling bifurcation and further by transition to chaotic motion. This scenarios repeats from 1480 rpm

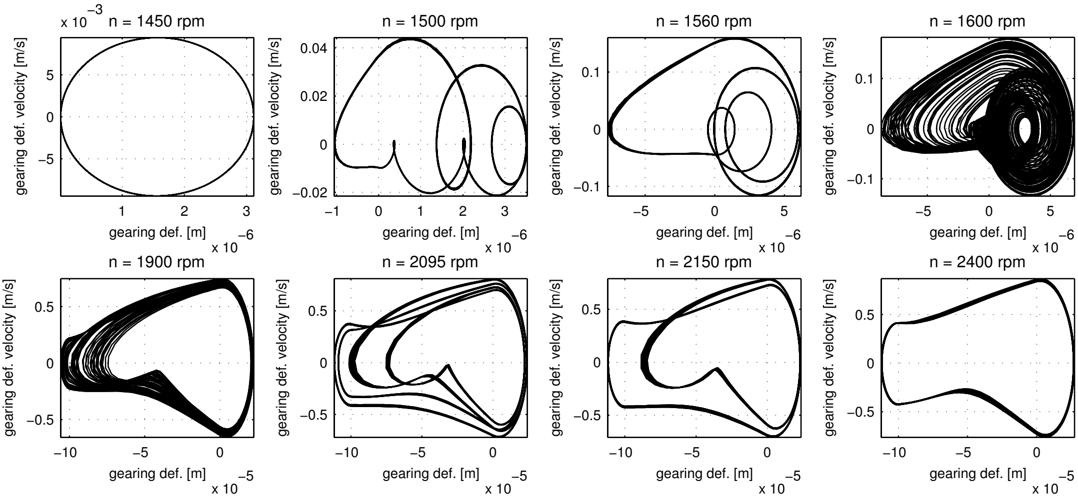


**Fig. 3** Examples of phase trajectories of gearing deformation $d_z$ in gearing

up to 1870 rpm. After that, a further amplitude jump appears which signifies a transition to both sided impact motion within the clearance of the gear mesh. The chaotic impact motion changes to quasi-periodic motion at 2100 rpm and then after reaching the value of 2360, the motion becomes again periodic. To have a clearer view to the behavior represented by bifurcation diagram, phase trajectories of gearing deformation for chosen rotational speeds are plotted in fig. 3.

## 5 CONCLUSIONS

The general approaches for non-smooth systems were discussed in context of the impact motion. The simple system with clearance was mentioned as an example of the large class of problems. The particular numerical realization was implemented for the test gear drive and numerical solution of motion of the system was accomplished. In this case, the smoothing method is the most efficient and switch method is the most inefficient one. The difference between analytical approximation of Heaviside function by arcustangens function and direct use of Heaviside function in MATLAB was shown. There is the same number of integration steps as in case of switch model but the computational time is lower. The bifurcation diagram shows how the quality of oscillations changes with changing of rotational speed of the drive and gives important information about impact vibrations.

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