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MODELLING OF $E_2/E_2/1/m$ QUEUEING SYSTEM SUBJECT TO BREAKDOWNS AND CUSTOMER REJECTION DURING FAILURE

MODELOVÁNÍ $E_2/E_2/1/m$ SYSTÉMU HROMADNÉ OBSLUHY PODLÉHAJÍCÍHO PORUCHÁM A S ODMÍTÁNÍM ZÁKAZNÍKŮ PŘI PORUŠE

Abstract

The paper deals with modelling of a finite single-server queueing system with a server subject to breakdowns. Customer interarrival times and customer service times follow the Erlang distribution defined by the shape parameter k=2 and the scale parameter 2λ or 2μ respectively. It is considered that server failures can occur when the server is either idle or busy – operate-independent failures. Further it is assumed that the system empties when the server is broken (all customers are being rejected). Random variables relevant to server failures and repairs are exponentially distributed. In the paper the state transition diagram is shown, the system of linear equations describing the system in the steady state and the formulas for several performance measures computation are presented. At the end of the paper there are shown some graphical dependencies.

Abstrakt

Článek je věnován modelování konečného jednolinkového systému hromadné obsluhy s linkou podléhající poruchám. Časové intervaly mezi příchody zákazníků k systému a doby obsluhy zákazníků se řídí Erlangovým rozdělením s parametrem tvaru k = 2 a parametrem měřítka 2λ, resp. 2μ. Je uvažováno, že poruchy obslužné linky mohou nastat, buď když linka nepracuje nebo když obsluhuje zákazníka – poruchy jsou tedy nezávislé na provozu linky. Dále je uvažováno, že systém je prázdný, když je linka v poruše (všichni zákazníci jsou odmítáni). Náhodné proměnné vztahující se k poruchám a opravám linky jsou rozděleny exponenciálně. V článku je uveden přechodový graf, soustavu lineárních rovnic popisujících systém ve stacionárním stavu a vztahy pro výpočet vybraných provozních charakteristik. V závěru příspěvku jsou uvedeny grafické závislosti.

1 INTRODUCTION

Queueing systems represent a lot of practical systems which can be found in technical practice, such as manufacturing, computer, telecommunication or transport systems. As can be seen in many books devoted to queueing theory, such as Cooper R.B. [1] or Bolch G. et al. [2], in most common queueing models the fact that a server is subject to failures is often ignored. However in technical practice this fact has to be often considered because server failures may adversely affect performance measures of a studied queueing system. Such type of queueing system has to be modelled as an unreliable queueing system in which the server is successively failure-free and broken.

The attention will be paid to a mathematical model of a finite single-server queueing system with the server subject to breakdowns, where customer interarrival times and service times follow the

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Erlang distribution. Further it is assumed that times between failures and times to repair follow the exponential distribution. The model presented in the paper is a modification of the model presented in the paper [3]. The essential difference lies in the following fact. In the model published formerly it is assumed the costumer which service has been interrupted goes back to the queue or leave the system when there is no free place in the system. However, in the model presented in the paper it is assumed that the system empties when the server is broken, that means all customers are being rejected during server failures (by other words it can be said that the system is closed). The situation can arise in practice for example when time to repair is too long and customers are not willing to wait and therefore leave the system to satisfy its request somewhere else.

2 ASSUMPTIONS OF THE MODEL

Let us study a single server queueing system with a finite capacity equal to m, where $m>1$, that means there are in total m places for customers in the system – single place in the service and m-1 places intended for waiting of customers. Let us consider that customers are served one by one according to the FCFS service discipline.

Let customer interarrival times follow the Erlang distribution with the shape parameter $k = 2$ and the scale parameter 2λ; therefore the mean interarrival time is equal to $\frac{2}{2\lambda} = \frac{1}{\lambda}$. Costumer service times are an Erlang random variable with the shape parameter k=2 as well, but with the scale parameter 2μ; thus the mean service time is equal to $\frac{2}{2\mu} = \frac{1}{\mu}$. The value of the shape parameter is assumed equal to 2 in order not to complicate mathematical model. However, the model can be easily extended for greater values of the shape parameter.

Let us assume that the server is successively failure-free (or available we can say) and broken. It is assumed that failures of the server can occur when the server is either idle or busy – server failures are operate-independent. Let us consider that times between failures are an exponential random variable with the parameter η, thus the mean time between failures is equal to $\frac{1}{\eta}$. Let times to repair be an exponential random variable as well, but with the parameter ξ; the mean time to repair is therefore equal to $\frac{1}{\xi}$. It is obvious that the server steady state availability A (the fraction of time the server is available) is equal to:

$$A = \frac{\frac{1}{\eta}}{\frac{1}{\eta} + \frac{1}{\xi}} = \frac{\xi}{\eta + \xi} \tag{1}$$

and the server steady-state unavailability U (the fraction of time the server is broken) is:

$$U = 1 - A = \frac{\eta}{\eta + \xi}\,. \tag{2}$$

As regards behavior of customers at the moment of the server failure, it is considered that all the customers finding in the system are rejected and all the customers incoming during repairing of the server are rejected as well; by others words the system empties when the server is down.

2 MATHEMATICAL MODEL

The studied queueing system can be modeled by method of stages. The method exploits the fact that the Erlang distribution with the shape parameter k and the scale parameter denoted as $k\lambda$ or $k\mu$ is the sum of the k independent exponential distribution with the same parameter $k\lambda$ or $k\mu$. Therefore the system can be modeled by using Markov chains.

Let us divide the states of the system into two groups:

- The failure-free system states are denoted by the triplet *k*,*v*,*o*, where the representation of the particular numbers in the triplet is:
    - the number *k* represents the number of the customers finding in the system, where $k \in \{0,1,..., m\}$,
    - the number *v* represents the terminated phase of the customer arrival, where $v \in \{0,1\}$,
    - the number *o* represents the terminated phase of the customer service, where $o \in \{0,1\}$.
- The states in which the server is broken are denoted by the notation P0,*v*, where:
    - the mark P0 expresses the fact that the server is broken and there is no customer finding in the system,
    - the number *v* represents the terminated phase of the customer arrival, where $v \in \{0,1\}$.

Let us illustrate the studied queueing system graphically as a state transition diagram; the diagram is shown in figure 1. The vertices represent the individual system states and the oriented edges indicate the possible transitions with the corresponding rate.

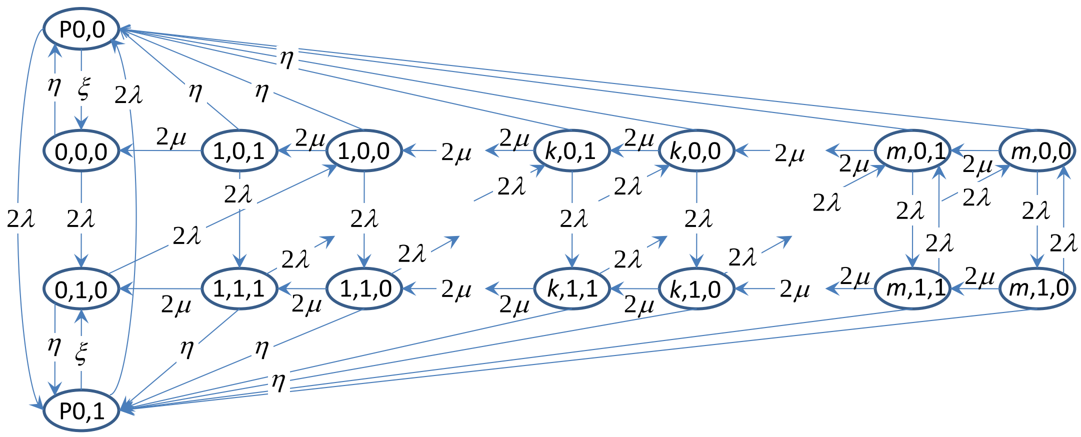

Fig. 1 The state transition diagram.

On the basis of the state transition diagram the finite system of the differential equations can be obtained for the probabilities of the individual states depending on the time *t*. But in practice the system is usually investigated in steady state for $t \to \infty$, thus the finite system of the linear equations has the following form:

$$(2\lambda + \eta)P_{0,0,0} = 2\mu P_{1,0,1} + \xi P_{P0,0}, \tag{3}$$

$$(2\lambda + \eta)P_{0,1,0} = 2\lambda P_{0,0,0} + 2\mu P_{1,1,1} + \xi P_{P0,1}, \tag{4}$$

$$(2\lambda + 2\mu + \eta)P_{1,0,1} = 2\mu P_{1,0,0}, \tag{5}$$

$$(2\lambda + 2\mu + \eta)P_{k,1,1} = 2\lambda P_{k,0,1} + 2\mu P_{k,1,0} \text{ for } k = 1,2,..., m, \tag{6}$$

$$(2\lambda + 2\mu + \eta)P_{k,0,0} = 2\lambda P_{k-1,1,0} + 2\mu P_{k+1,0,1} \text{ for } k = 1,2,..., m-1, \tag{7}$$

$$(2\lambda+2\mu+\eta)P_{k,1,0}=2\lambda P_{k,0,0}+2\mu P_{k+1,1,1} \text{ for } k=1,2,...,m-1, \tag{8}$$

$$(2\lambda+2\mu+\eta)P_{k,0,1}=2\lambda P_{k-1,1,1}+2\mu P_{k,0,0} \text{ for } k=2,3,...,m-1, \tag{9}$$

$$(2\lambda+2\mu+\eta)P_{m,0,1}=2\lambda P_{m-1,1,1}+2\lambda P_{m,1,1}+2\mu P_{m,0,0}, \tag{10}$$

$$(2\lambda+2\mu+\eta)P_{m,0,0}=2\lambda P_{m-1,1,0}+2\lambda P_{m,1,0}, \tag{11}$$

$$(2\lambda+2\mu+\eta)P_{m,1,0}=2\lambda P_{m,0,0}, \tag{12}$$

$$(2\lambda+\xi)P_{P0,0}=\eta P_{0,0,0}+\eta\sum_{k=1}^{m}\sum_{o=0}^{1}P_{k,0,o}+2\lambda P_{P0,1}, \tag{13}$$

$$(2\lambda+\xi)P_{P0,1}=\eta P_{0,1,0}+\eta\sum_{k=1}^{m}\sum_{o=0}^{1}P_{k,1,o}+2\lambda P_{P0,0}. \tag{14}$$

Clearly, for the steady state probabilities the normalization equation must hold:

$$P_{0,0,0}+P_{0,1,0}+\sum_{k=1}^{m}\sum_{v=0}^{1}\sum_{o=0}^{1}P_{k,v,o}+P_{P0,0}+P_{P0,1}=1. \tag{15}$$

It should be noticed that equation (14) is linear combination of equations (3) up to (13), therefore equation (14) is omitted and replaced by normalization equation (15) to solve the system. By solving the linear equation system formed from equations (3) – (13) and (15) the stationary probabilities of the particular states of the system can be computed. The probabilities are needed for performance measures computing.

Let us consider three performance measures – the mean number of the customers in the service *ES*, the mean number of the waiting customers *EL* and the mean number of the broken servers *EP*. All of them can be computed according to the formula for the mean value computation of a discrete random variable, where successively the random variable $S\in\{0,1\}$ is the number of the costumers in the service, the variable $L\in\{0,m-1\}$ the number of the waiting customers and the variable $P\in\{0,1\}$ the number of the broken servers.

The mean number of the costumers in the service *ES* can be computed:

$$ES=\sum_{k=1}^{m}\sum_{v=0}^{1}\sum_{o=0}^{1}P_{k,v,o}, \tag{16}$$

for the mean number of the waiting customers *EL* it can be written:

$$EL=\sum_{k=2}^{m}(k-1)\sum_{v=0}^{1}\sum_{o=0}^{1}P_{k,v,o}, \tag{17}$$

and finally the mean number of the broken servers *EP* can be expressed by the formula:

$$EP=P_{P0,0}+P_{P0,1}. \tag{18}$$

On the basis of formulas (1) and (16) and formulas (2) and (18) it is obvious that:

$$P_{0,0,0}+P_{0,1,0}+\sum_{k=1}^{m}\sum_{v=0}^{1}\sum_{o=0}^{1}P_{k,v,o}=\frac{\xi}{\eta+\xi} \tag{19}$$

and

$$P_{P0,0}+P_{P0,1}=\frac{\eta}{\eta+\xi}. \tag{20}$$

4 EXECUTED NUMERICAL EXPERIMENTS

Let us consider the studied queueing system with 5 places in the system. In table 1 the values of applied random variables parameters are summarized.

Tab. 1 The applied random variables parameters.

Random variable (RV)	Applied parameters of RV
Inter-arrival times – Erlang RV	$k = 2$; $2\lambda = 18$ h^{-1}
Service times – Erlang RV	$k = 2$; $2\mu = 20$ h^{-1}
Times between failures – exponential RV	$\eta = 200^{-1}, 190^{-1}, \ldots, 20^{-1}, 10^{-1}$ h^{-1}
Times to repair – exponential RV	$\xi = 0.2$ h^{-1} – 1st experiment $\xi = 0.1$ h^{-1} – 2nd experiment

There were executed two experiments differing in the value of the parameter ξ. For each value of the parameter η the stationary probabilities were computed numerically by using software Matlab. On the basis of the stationary probabilities knowledge the performance measures can be computed according to the corresponding formulas. Let us focus our attention on the performance measures *ES*, *EL* and *EP*. The dependencies of the individual performance measures on the reciprocal value of the parameter η are shown in figures 2, 3 and 4.

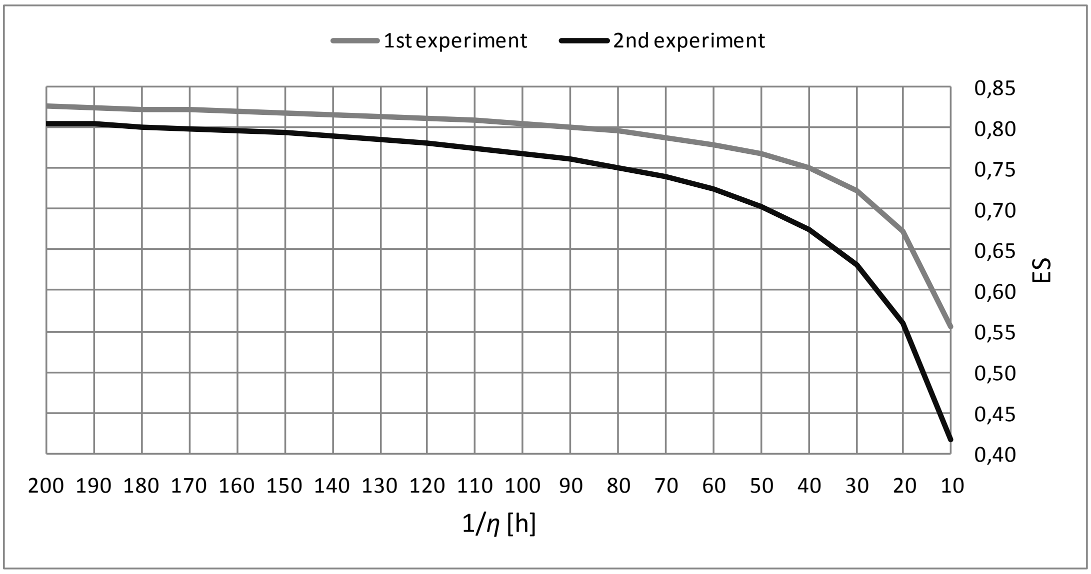

Fig. 2 The dependence of *ES* on the parameter $1/\eta$.

As can be seen in figure 2, increasing value of the parameter η (or decreasing value of the reciprocal value of the parameter η) causes decreasing of the mean number of the customers in the system *ES*. This fact could be logically expected because more frequent failures mean lower fraction of the time in which the server is able to serve incoming customers.

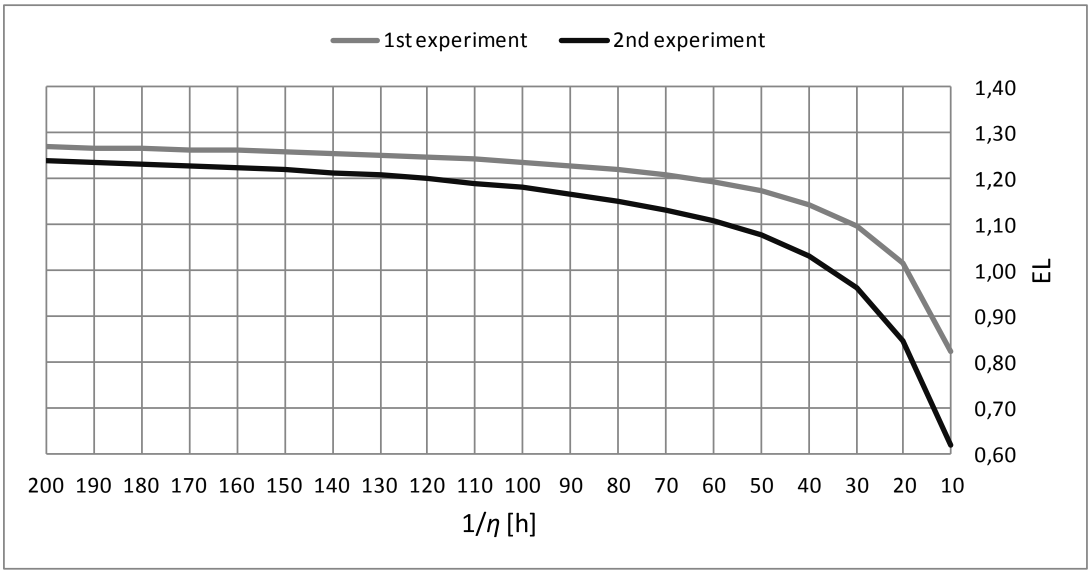

Fig. 3 The dependence of *EL* on the parameter 1/*η*.

In figure 3 the decreasing dependency can be seen due to the fact that the system empties when the server is broken; that means the stationary probability the system is empty increases with increasing the value of the parameter *η*.

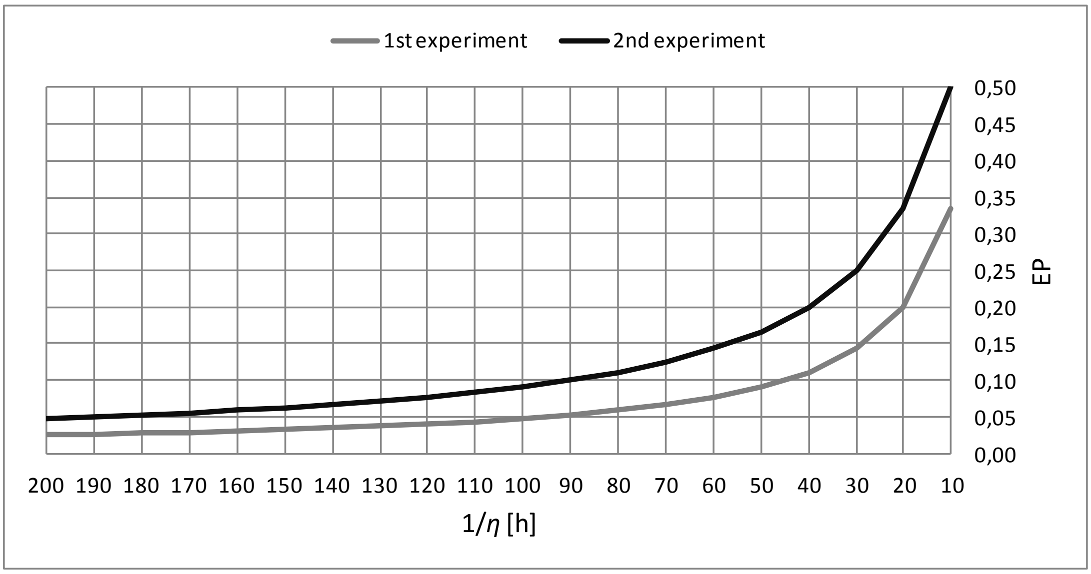

Fig. 4 The dependence of *EP* on the parameter 1/*η*.

As can be seen in figure 4, the dependency of the performance measure *EP* is increasing. This fact is obvious as well. The reached outcomes can be easily verified because for the performance measure *EP* formula (20) must hold. For example, in the case of 2nd experiment for $\eta = 10^{-1}\ h^{-1}$ the mean number of broken servers is equal to 0.5, the value is in accordance with formula (20).

5 CONCLUSIONS

In this paper we paid our attention on the finite $E_2/E_2/1/m$ queue with the server subject to breakdowns. We considered that server failures are operate-independent. Further we assumed that the system empties while the server is being repaired; therefore all customers are rejected when the server is down.

For the studied system we developed the state transition diagram and wrote the system of the linear equations for the steady state. The stationary probabilities can be computed numerically for example by using Matlab. When we know the probabilities, we are able to compute several performance measures we are interested in. Further we presented some numerical experiments executed with the model; on the basis of them we got some graphical dependencies of the selected performance measures on the reciprocal value of the parameter $1/\eta$.

In the future we would like to find the formula for the customer loss probability, because this performance measures is often very important for finite queueing systems.

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