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MODELLING AND SIMULATION OF UNRELIABLE $E_2/E_2/1/m$ QUEUEING SYSTEM

MODELOVÁNÍ A SIMULACE NESPOLEHLIVÉHO $E_2/E_2/1/m$ SYSTÉMU HROMADNÉ OBSLUHY

Abstract

This paper is devoted to modelling and simulation of an $E_2/E_2/1/m$ queueing system with a server subject to breakdowns. The paper introduces a mathematical model of the studied system and a simulation model created by using software CPN Tools, which is intended for modelling and a simulation of coloured Petri nets. At the end of the paper the outcomes which were reached by both approaches are statistically evaluated.

Abstrakt

Článek je věnován modelování a simulaci $E_2/E_2/1/m$ systému hromadné obsluhy S obslužnou linkou podléhající poruchám. Příspěvek představuje matematický model studovaného systému a simulační model vytvořený S využitím software CPN Tools, který je určen pro modelování a simulaci barevných Petriho sítí. V závěru článku jsou výsledky dosažené oběma přístupy statisticky vyhodnoceny.

1 INTRODUCTION

Server working without its failures is usually assumed in the queueing theory (see for example in [1], [2] or [3]). Server failures can be neglected if they are not so frequent. In case of more frequented failures the impact of server failures should not be neglected. Many authors studied a behavior of the diverse unreliable queueing systems. Most of them investigated mathematically the simplest queueing models – Markovian queueing systems – or mathematically the most difficult queueing models with general distribution of costumers inter-arrival times, service times etc. For example paper [4] is devoted to modelling of the unreliable $M/M/1/m$ queueing system, paper [5] introduces a model of the unreliable $M/M/1/\infty$ queueing system with impatient costumers, papers [6] a [7] are focused on unreliable $M/M/1/\infty$ queueing systems with failures causing departure of all customers finding in the system. Some unreliable $M/G/1/\infty$ queueing systems are studied in paper [8]. In work [9] Mean value approach is applied for computation the mean number of customers in the system EK and the mean sojourn time of customer in the system ET for unreliable $M/M/1/\infty$ and $M/G/1/\infty$ queueing systems. The models of the queueing systems with Erlang customers inter-arrival times or Erlang service times (these queueing systems are sometimes called Semimarkovian) are not so common. This paper presents a mathematical and a simulation model of $E_2/E_2/1/m$ queueing system with an unreliable server.

Let assume the queueing system with a single unreliable server. Incoming customers wait for the service in a queue with capacity of $m-1$ customers. Thus there are m places in the system.

Let consider that a server breakdown can occur at any time. That means the server can break if it is busy (a costumer is in the service) or idle (there is no customer in the service). Notice that we do not consider a possibility of the several failures occurrence at the same time. That means that after

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occurrence of a breakdown another failure cannot occur until the server is broken (by other words failures which are incoming during server repairing are rejected).

After occurrence of the breakdown two different events can occur (if there is a costumer in the service just at the moment):

- If there are less than m–1 customers in the queue, costumer comes back to the queue and his service starts from the beginning after that.
- If there are exactly m–1 customers in the queue, costumer leaves the system and customer is considered to be rejected.

Customers inter-arrival times follow Erlang distribution with shape parameter $k = 2$ and scale parameter 2λ; the mean value is then equal to $\frac{2}{2\lambda} = \frac{1}{\lambda}$. Failures occur according to the Poisson process with rate $\bar{\lambda}$. The time between failures is an exponential random variable with mean value equal to $\frac{1}{\bar{\lambda}}$.

Costumer service time is Erlang random variable with shape parameter $k = 2$ and scale parameter 2μ; thus the mean service time is equal to $\frac{2}{2\mu} = \frac{1}{\mu}$. Server repair time is exponential random variable with parameter $\bar{\mu}$, mean server repair time is $\frac{1}{\bar{\mu}}$.

Customers are served one by one according to FIFO (First In - First Out) discipline. On the basis of the assumptions we can say that the presented system is according to Kendall's notation $E_2/E_2/1/m$ system with an unreliable server.

2 MATHEMATICAL MODEL

The queueing system can be modeled by Method of stages (see for example [9]). The method exploits the fact that Erlang distribution with shape parameter K and scale parameter denoted as $k\lambda$ or $k\mu$ is sum of K independent exponential distribution with the same parameter $k\lambda$ or $k\mu$.

The states of the studied system can be divided into two groups:

- states of the systems in which the server is failure-free, the states are denoted by notation k,v,o, where:
  - k represents a number of the customers finding in the system,
  - v represents a terminated phase of the customer arrival,
  - o represents a terminated phase of the customer service.
- States of server failure are denoted by the notation Pk,v, where:
  - A letter P expresses the server failure,
  - k represents a number of the customers finding in the system,
  - v represents a terminated phase of the customer arrival.

Let illustrate the queueing model graphically as a state transition diagram (see in Fig. 1). The vertices represent the states of the particular system and oriented edges indicate the possible transitions with corresponding rate. Notice that the diagram in fig. 1 is drawn without loops.

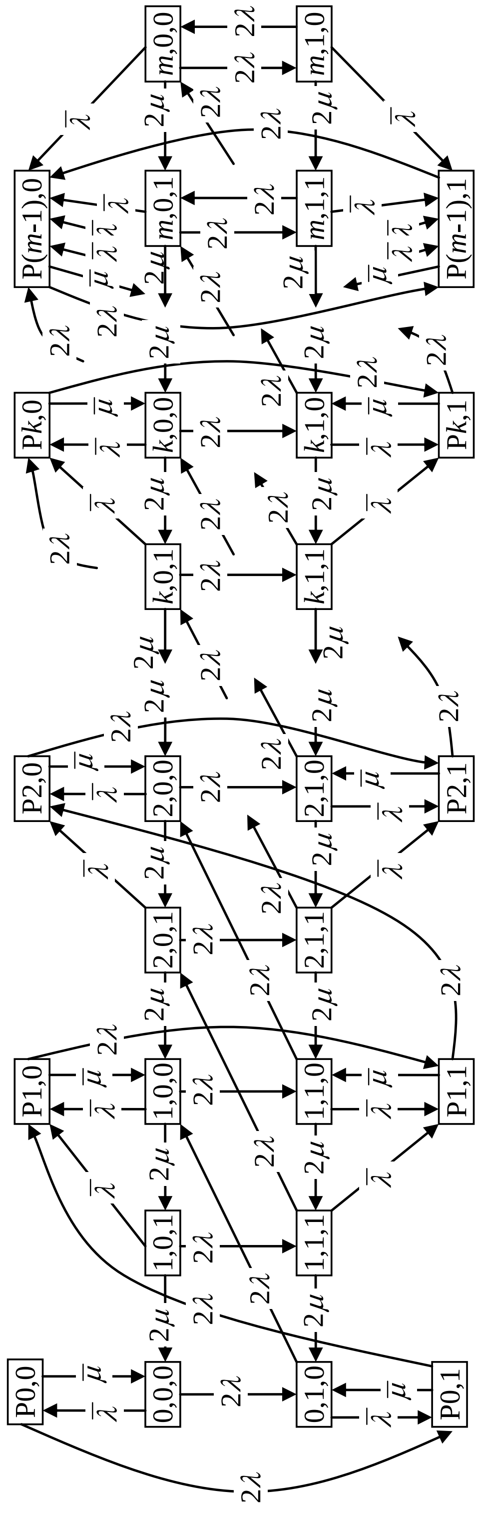

Fig. 1 State transition diagram of unreliable $E_2/E_2/1/m$ queueing system.

On the basis of the state transition diagram we can obtain finite system of the differential equations for probabilities of the particular states depending on time *t*. For $t \to \infty$ we get the system of the linear equations for steady state probabilities that are not dependent on time *t*:

$$0 = -\left(2\lambda + \bar{\lambda}\right)P_{0,0,0} + 2\mu P_{1,0,1} + \bar{\mu} P_{P0,0} ,$$
$$0 = 2\lambda P_{0,0,0} - \left(2\lambda + \bar{\lambda}\right)P_{0,1,0} + 2\mu P_{1,1,1} + \bar{\mu} P_{P0,1} ,$$
$$0 = -\left(2\lambda + 2\mu + \bar{\lambda}\right)P_{1,0,1} + 2\mu P_{1,0,0} ,$$
$$0 = 2\lambda P_{k,0,1} - \left(2\lambda + 2\mu + \bar{\lambda}\right)P_{k,1,1} + 2\mu P_{k,1,0} \text{ for } k = 1,..., m ,$$
$$0 = 2\lambda P_{k-1,1,0} - \left(2\lambda + 2\mu + \bar{\lambda}\right)P_{k,0,0} + 2\mu P_{k+1,0,1} + \bar{\mu} P_{Pk,0} \text{ for } k = 1,..., m-1 ,$$
$$0 = 2\lambda P_{k,0,0} - \left(2\lambda + 2\mu + \bar{\lambda}\right)P_{k,1,0} + 2\mu P_{k+1,1,1} + \bar{\mu} P_{Pk,1} \text{ for } k = 1,..., m-1 ,$$
$$0 = 2\lambda P_{k-1,1,1} - \left(2\lambda + 2\mu + \bar{\lambda}\right)P_{k,0,1} + 2\mu P_{k,0,0} \text{ for } k = 2,..., m-1 ,$$
$$0 = 2\lambda P_{m-1,1,1} - \left(2\lambda + 2\mu + \bar{\lambda}\right)P_{m,0,1} + 2\lambda P_{m,1,1} + 2\mu P_{m,0,0} ,$$
$$0 = 2\lambda P_{m-1,1,0} - \left(2\lambda + 2\mu + \bar{\lambda}\right)P_{m,0,0} + 2\lambda P_{m,1,0} ,$$
$$0 = 2\lambda P_{m,0,0} - \left(2\lambda + 2\mu + \bar{\lambda}\right)P_{m,1,0} ,$$
$$0 = \bar{\lambda} P_{0,0,0} - \left(2\lambda + \bar{\mu}\right)P_{P0,0} ,$$
$$0 = \bar{\lambda} P_{0,1,0} + 2\lambda P_{P0,0} - \left(2\lambda + \bar{\mu}\right)P_{P0,1} ,$$
$$0 = \bar{\lambda} P_{k,0,1} + \bar{\lambda} P_{k,0,0} + 2\lambda P_{P(k-1),1} - \left(2\lambda + \bar{\mu}\right)P_{Pk,0} \text{ for } k = 1,..., m-2 ,$$
$$0 = \bar{\lambda} P_{k,1,1} + \bar{\lambda} P_{k,1,0} + 2\lambda P_{Pk,0} - \left(2\lambda + \bar{\mu}\right)P_{Pk,1} \text{ for } k = 1,..., m-2 ,$$
$$0 = \bar{\lambda} P_{m-1,0,1} + \bar{\lambda} P_{m-1,0,0} + \bar{\lambda} P_{m,0,1} + \bar{\lambda} P_{m,0,0} + 2\lambda P_{P(m-2),1} - \left(2\lambda + \bar{\mu}\right)P_{P(m-1),0} + 2\lambda P_{P(m-1),1} ,$$
$$0 = \bar{\lambda} P_{m-1,1,1} + \bar{\lambda} P_{m-1,1,0} + \bar{\lambda} P_{m,1,1} + \bar{\lambda} P_{m,1,0} + 2\lambda P_{P(m-1),0} - \left(2\lambda + \bar{\mu}\right)P_{P(m-1),1}$$

including the condition $\sum_{k=0}^{m}\sum_{v=0}^{1}\sum_{o=0}^{1} P_{k,v,o} + \sum_{k=0}^{m-1}\sum_{v=0}^{1} P_{Pk,v} = 1$. (1)

By solving of this linear equations system (1) we get stationary probabilities of the particular states of the system that are needed for performance measures computing of studied queueing system.

Let consider three selected performance measures – the mean number of the customers in the service *ES*, the mean number of the waiting customers *EL* and the mean number of the servers in failure *EP*. All of these performance measures can be computed according to the formula for mean value of discrete random variable computation.

For the mean number of the costumers in the service *ES* we can write:

$$ES = \sum_{k=1}^{m}\sum_{v=0}^{1}\sum_{o=0}^{1} P_{k,v,o} . \tag{2}$$

Notice that in the case of a single server queueing system the mean number of the customers in the service is equal to the server utilization usually denoted as κ and also represents the fraction of the time when the server is busy.

The mean number of the waiting costumers *EL* can be expressed by the formula:

$$EL = \sum_{k=2}^{m}\left(k-1\right)\sum_{v=0}^{1}\sum_{o=0}^{1} P_{k,v,o} + \sum_{k=1}^{m-1} k \sum_{v=0}^{1} P_{Pk,v} . \tag{3}$$

And finally for the mean number of the servers in failure *EP* (this performance measure can be also interpreted as the fraction of the time when the server is broken) we get:

$$EP = \sum_{k=0}^{m-1}\sum_{v=0}^{1} P_{Pk,v} . \tag{4}$$

3 PETRI NET MODEL

To validate outcomes which were reached by above-mentioned mathematical model solution Petri net model of studied queueing system was created by using CPN Tools – Version 2.2.0. Software CPN Tools is designed for editing, simulating and analyzing coloured Petri nets. The model is compound of 14 places and 12 transitions. The presented Petri net models unreliable $E_2/E_2/1/5$ queueing system fulfilling the conditions mentioned in chapter 2, values of random variable parameters are shown in tab. 1.

Tab. 1 Values of random variable parameters.

Random variable (RV)	Random variable parameters
Customers inter-arrival times – Erlang RV	$k = 2.2\lambda = 18h^{-1}$
Service times – Erlang RV	$k = 2.2\mu = 20h^{-1}$
Breakdown inter-arrival times – Exponential RV	$\bar{\lambda} = 0.02h^{-1}$
Repair times – Exponential RV	$\bar{\mu} = 0.2h^{-1}$

Concrete values of random variables are generated during a simulation through defined function fun ET(k, mi) = round(erlang(k, mi/3600.0)), thus a second is the applied unit of time. Notice that the values from exponential distribution can be generated through this function as well, because exponential distribution is the special case of Erlang distribution with shape parameter K = 1. There were executed 30 experiments; each experiment was terminated after a million steps (a step corresponds to a transition firing).

Created coloured Petri net uses 3 token colours – tokens colour c represent customers, tokens colour f model the server failures and the auxiliary tokens colour p model for example free queue places, free servers etc. Tokens c and f are timed.

The place called "Customer" with initial marking c and the transition "Customer initialization" model arrival process of the customers, all tokens leaving this transition are labeled by corresponding time stamp to model Erlang inter-arrival times. a customer approaching to the queueing system finding in the place "Input place for customer" can come in the queue if there is a token p in the place "Free queue places" (initial marking of the place is equal to the queue capacity – 4`p), otherwise the customer is rejected. The transition "Customer rejection" is enabled only if there are exactly 4 customers in the queue because the transition is connected with the place "Queue" by a testing arc with arc expression 4`c. The transition "Input into service" is enabled if there is a customer in the queue and the server is free (that means the server is idle and failure-free; the place "Free servers" with initial marking p models available servers). In the place "Service" there are placed tokens modeling service of customers, the appropriate service time is ensured by the time stamp update through the transition "Input into service" firing. The transition "Output" is enabled if there is a customer in the service and the server is working (there is a token p in the auxiliary place "Working servers").

Arrival process of the server failures is modeled analogously like the arrival process of customers by the place "Breakdown" with initial marking e and the transition "Breakdown initialization". The time stamp of all tokens leaving the transition is increased by a value arising out of exponential distribution with appropriate parameter. The transition "Breakdown rejection" is enabled only if the server is already broken – the transition is connected by a testing arc with the auxiliary place "Broken servers", the arc expression is equal to p. The transition "Idle server breakdown" is enabled if the server is idle and failure-free – the auxiliary places "Failure-free servers" with initial marking p and "Free servers" with initial marking p are input places of the transition. The transition "Busy server breakdown" is enabled if the server is busy and failure-free (there is a token p in the place "Working servers" and in the place "Failure-free servers"). The customer service abortion due to the breakdown occurrence is modeled through the place "Service abortion" and the transition "Service abortion due to breakdown", firing of the transition "Busy server breakdown" causes an input of the customer to place "Input place for customer" according to assumptions of mathematical model. Firing of transition "Repair beginning" causes the time stamp update of a token f which models server breakdown, the server repair is terminated by firing the transition "Repair termination".

Created model in initial marking is shown in Fig. 2 below.

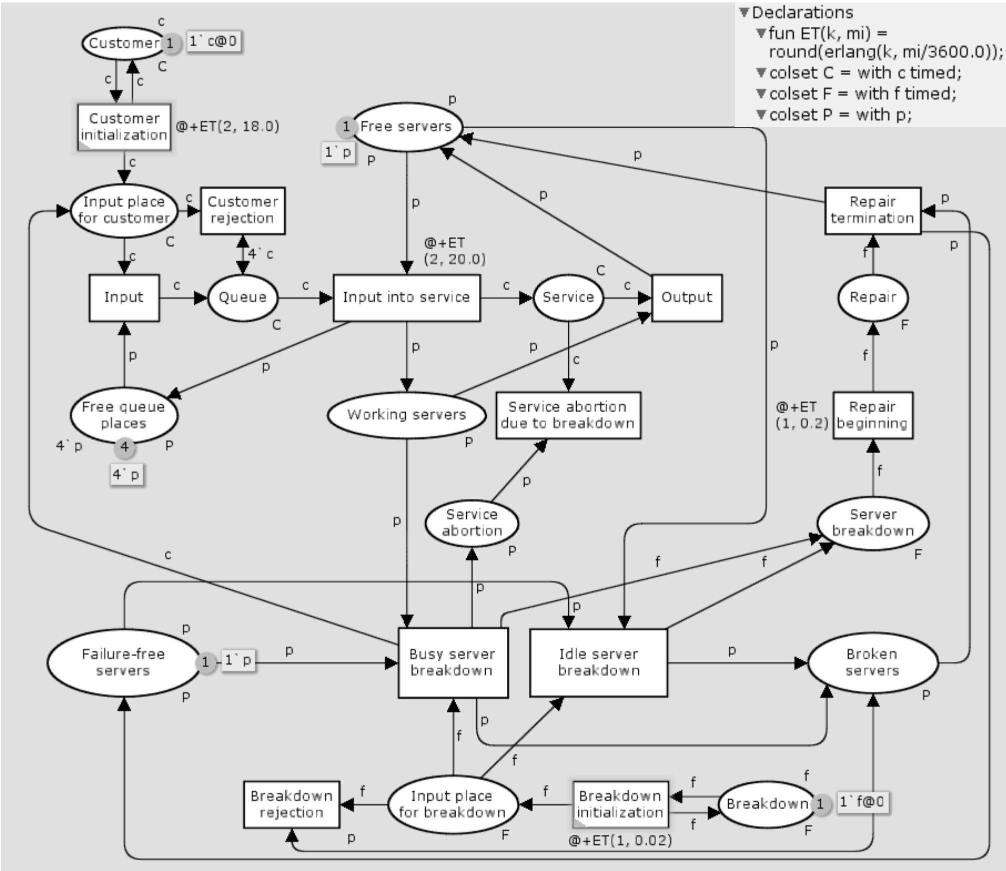

Fig. 2 Petri net model – initial marking

3 marking size monitoring functions were applied for computing selected performance measures during simulation:

- The monitorig function ES bonded with the place "Working servers" enables the mean number of the customers in the service estimation.
- The monitoring function EL is bonded with the place "Queue" and serves to the mean number of the waiting customers estimation.
- The monitoring function EP is associated with the place "Broken servers" and enables the mean number of the servers in failure estimation.

4 EVALUATION OF EXECUTED EXPERIMENTS

By simulation experiments three data sets (*ES*, *EL*, *EP*) with range $n = 30$ were obtained. With the view of the further statistical evaluation the normality of particular data sets was tested using χ^2 goodness-of-fit test; the outcomes are shown in tab. 2. As for all data sets p-value is greater than 0,05, we do not reject particular hypotheses about normal distribution. Notice that all statistical computations were executed using software Statgraphics plus 5.0.

Tab. 2 Outcomes of goodness-of-fit tests.

Normal distribution parameters	P-value
$\mu \doteq 0.77237, \sigma \doteq 0.00396$	0.61596
$\mu \doteq 1.56313, \sigma \doteq 0.01202$	0.91608
$\mu \doteq 0.09089, \sigma \doteq 0.00431$	0.52892

Let focus on the mean value confidence intervals for particular data sets and testing the hypothesis the mean value of single performance measure gained by simulation experiments is equal to the value gained by mathematical model solution in steady state; let assume $\alpha = 0.05$ and the alternative hypothesis is in the form not equal. Stationary probabilities were found numerically using software Matlab as the solution of the finite linear equations system (1); notice that the penult equation was dropped. The performance measures were further computed according formulas (2), (3) and (4). Reached outcomes are summarized in table 3.

Tab. 3 Comparison of analytic and simulation outcomes

Data set	Value gained numerically	Confidence interval	P-value
ES	0.77241	(0.77089; 0.77385)	0,95175
EL	1.56324	(1.55865; 1.56762)	0.96148
EP	0.09091	(0.08927; 0.09250)	0.97496

On the basis of reached p-values we can state that there are no statistical significant differences between studied performance measures gained numerically in steady state and mean values of relevant performance measures gained simulation experiments.

5 CONCLUSIONS

This paper presents two models of $E_2/E_2/1/m$ queueing system with a server subject to breakdowns – mathematical model created by using Method of stages and coloured Petri net simulation model created through CPN Tools. The major part of the paper is focused on mathematical models of the studied system. At the end of the paper the reached outcomes for $m = 5$ and concrete parameters of random variables used in the queueing system model are evaluated.

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