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DESCRIPTION AND APPLICATION OF THE CE-SE SCHEME

POPIS A APLIKACE CE-SE SCHÉMATU

Abstract

The CE-SE scheme is new numerical methodology for conservation laws. It was developed by Dr. Chang of NASA Glenn Research Center and his collaborators. The 1D and 2D variant of CE-SE scheme for scalar convection and for Euler equations for incompressible flows is described. The CE-SE scheme is compared with classical schemes in 1D. The solution to inviscid incompressible flow in GAMM channel in 2D is presented.

Abstrakt

CE-SE schéma je nová numerická metoda pro řešení zákonů zachování. V článku je popsána 1D a 2D varianta CE-SE schématu pro skalární rovnici a pro Eulerovy rovnice popisující nestlačitelné proudění. CE-SE schéma je srovnáváno s klasickými schématy v 1D a ve 2D je prezentováno řešení nevazkého, nestlačitelného proudění v GAMM kanále.

1 INTRODUCTION

The CE-SE (Conservation Element – Solution Element) scheme is new numerical methodology for conservation laws. It was developed by Dr. Chang of NASA Glenn Research Center and collaborators. The concept and methodology of this method are significantly different from those in the well-established traditional numerical methods. The flux is conserved in space and time. They are treated equally. The dependent variables and their derivatives are considered as individual unknowns to be solved simultaneously at each grid point [1-2].

2 1D AND 2D VERSIONS OF CE-SE SCHEME BE DESCRIBE

2.1 One-dimensional version be described

Consider the linear 1D unsteady advection equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \quad a \in R, \quad a \neq 0 \quad (1)$$

where constant a is the advection speed and u is an arbitrary scalar function. In the space-time Euclidean space E_2 , the integral form of (1) is

$$\oint_{\partial\Omega} \vec{h} \cdot d\vec{n} = 0 \quad (2)$$

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where $\vec{h} = (au, u)$ and $\partial\Omega$ is the boundary of arbitrary space-time region of Ω in E_2 , $d\vec{n} = \pm(dt, -dx)$ is normal vector of boundary $\partial\Omega$. The conservation element (CE) and solution element (SE) are the two basic elements. They are depicted in Fig. 1.

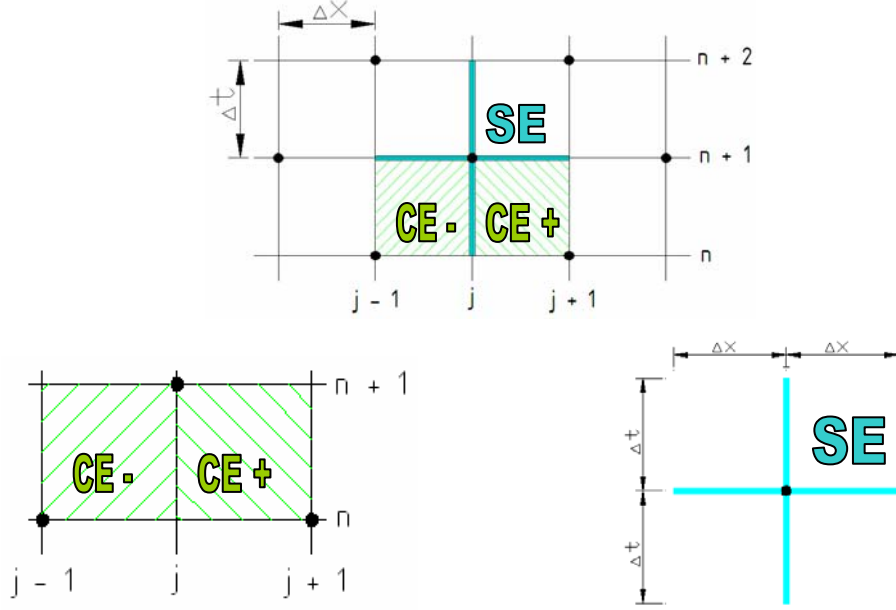


Fig. 1 Position of CE and SE elements in 1D

At each SE element $[j, n+1]$, $u(x,t)$ is approximated by the first-order Taylor series expansion:

$$u^*(x, t; j, n+1) = u_j^{n+1} + (u_x)_j^{n+1}(x - x_j) + (u_t)_j^{n+1}(t - t^{n+1}) \quad (3)$$

where, u_j^{n+1} , $(u_x)_j^{n+1}$ and $(u_t)_j^{n+1}$ are constant. After inserting of u^* to \vec{h} , we get $\vec{h} = (au^*, u^*)$.

The assumption that $u = u^*$ satisfies (1) implies $(u_t)_j^{n+1} = -a(u_x)_j^{n+1}$. We integrate equation (2) around $\Omega = \pm CE$ and we get two CE elements:

$$\frac{4}{\Delta x^2} F_{\pm}(j, n+1) = \pm \frac{1}{2} \left[(1 - \nu^2)(u_x)_j^{n+1} + (1 - \nu^2)(u_x)_{j\pm 1}^n \right] + \frac{2(1 \mp \nu)}{\Delta x} (u_j^{n+1} - u_{j\pm 1}^n) = 0 \quad (4)$$

where $\nu = (a\Delta t) / \Delta x$. From equation (4) we obtain two unknowns u_j^n and $(u_x)_j^n$:

$$u_j^{n+1} = \frac{1}{2} \left\{ (1 + \nu)u_{j-1}^n + (1 - \nu)u_{j+1}^n + (1 - \nu^2) \frac{\Delta x}{4} [(u_x)_{j-1}^n - (u_x)_{j+1}^n] \right\} \quad (5)$$

$$(u_x)_j^{n+1} = \frac{1}{2(1 - \nu^2)} \left\{ (\nu^2 - 1) \frac{4}{\Delta x} (u_{j-1}^n - u_{j+1}^n) - (1 - \nu^2) [(1 - \nu)(u_x)_{j-1}^n + (1 + \nu)(u_x)_{j+1}^n] \right\} \quad (6)$$

Stability condition is $\nu \leq 1$ see [2].

2.2 Two-dimensional version be described

Two-dimensional version is described at cartesian mesh. We study inviscid incompressible flow described by the set of Euler equations with artificial compressibility method. Those equations in matrix and quasi-linear form, can be written as:

$$W_t + \frac{\partial F}{\partial W} W_x + \frac{\partial G}{\partial W} W_y = 0 \quad (7)$$

$$W = [\tilde{p}, u, v], \quad F = [u, u^2 + \tilde{p}, uv], \quad G = [v, uv, v^2 + \tilde{p}], \quad \tilde{p} = \frac{p}{\rho}$$

where $\frac{\partial F}{\partial W}$ is jacobian's of flux F, $\frac{\partial G}{\partial W}$ is jacobian's of flux G, p is pressure, ρ is density

(constant) and u, v are velocities. From Eq.(7) we can get $W_{t,i,j}^n$:

$$W_t = -\frac{\partial F}{\partial W} W_x - \frac{\partial G}{\partial W} W_y \quad (8)$$

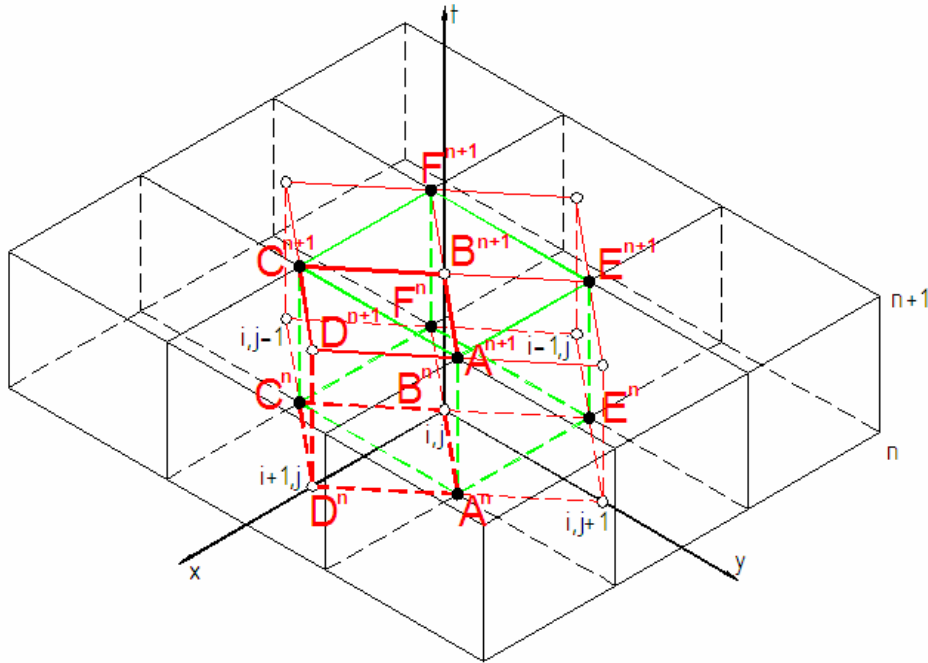


Fig. 2 Position of CE and SE elements in 2D

Fig. 2 shows four CE elements, one from four is $A^n B^n C^n D^n A^{n+1} B^{n+1} C^{n+1} D^{n+1}$ and the SE element is shown with black points $A^n E^n C^n F^n A^{n+1} E^{n+1} C^{n+1} F^{n+1}$. For SE element we get from Taylor series expansion in space-time:

$$(x, y, t) \in SE \Rightarrow W(x, y, t) = W_{i,j}^n + W_{x_{i,j}}^n (x - x_{i,j}) + W_{y_{i,j}}^n (y - y_{i,j}) + W_{t_{i,j}}^n (t - t^n) \quad (9)$$

For individual CE element we get from equation of conservation:

$$\begin{aligned} & \iint_{A^n B^n C^n D^n} W^n dx dy - \iint_{A^{n+1} B^{n+1} C^{n+1} D^{n+1}} W^{n+1} - \iint_{A^n B^n A^{n+1} B^{n+1}} \vec{n}_{AB}(F, G) - \iint_{B^n C^n B^{n+1} C^{n+1}} \vec{n}_{BC}(F, G) - \\ & \iint_{C^n D^n C^{n+1} D^{n+1}} \vec{n}_{CD}(F, G) - \iint_{D^n A^n D^{n+1} A^{n+1}} \vec{n}_{DA}(F, G) = 0 \end{aligned} \quad (10)$$

where \vec{n}_{AB} , \vec{n}_{BC} , \vec{n}_{CD} and \vec{n}_{DA} are normal vectors at appropriate for sides. We apply this equation at four CE elements around SE element. We get four values of W in time-step n+1, and then we get W in mesh point i, j and time-step n+1:

$$W_{i,j}^{n+1} = \frac{|CE 1|W 1 + |CE 2|W 2 + |CE 3|W 3 + |CE 4|W 4}{|CE 1| + |CE 2| + |CE 3| + |CE 4|} \quad (11)$$

Values of derivatives W_x^{n+1} and W_y^{n+1} in a SE element are calculated with least squares method.

3 COMPUTATIONAL RESULTS

3.1 One-dimensional scalar problem

We will solve initial-boundary value problem for scalar equation:

$$u_t + au_x = 0 \quad (12)$$

with initial condition (13), where $a=1$, $x \in (0;1)$, and boundary condition is $u(0,t) = \alpha(t) = 0$. We compare exact solution with two numericals schemes: first order upwind scheme and new CE-SE scheme for initial condition:

$$\begin{aligned} u(x,0) &= 0 \quad x \in (0,0.25) \cup (0.5,1) \\ u(x,0) &= (1 - \cos(8\pi x))/2 \quad x \in (0.25,0.5) \end{aligned} \quad (13)$$

Fig. 3 shows solution in time $t = 0.25$ and space step $\Delta x = 0.02$. Solution with upwind scheme doesn't produce any spurious oscillations, but is very inaccurate. Solution with CE-SE scheme is more accurate, but produces some oscillations in places with discontinuity in high derivatives.

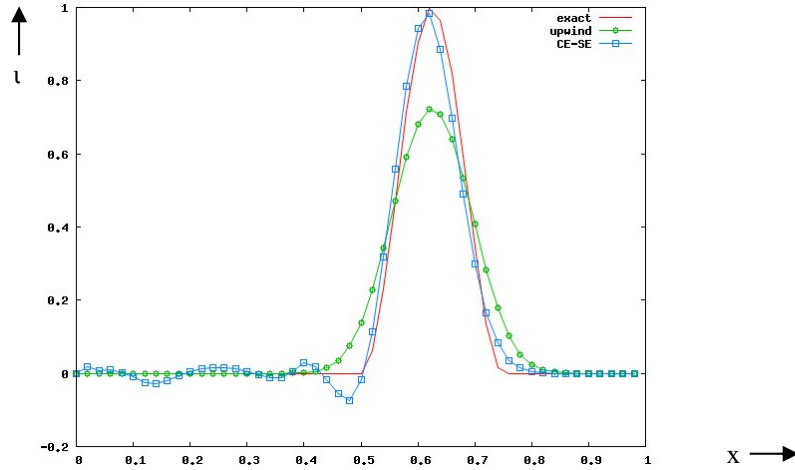


Fig. 3 Solution for first initial condition in time $t = 0.25$ and $\Delta x = 0.02$

3.2 Two-dimensional problem

We demonstrate of CE-SE method in two-dimensional problem on flow in GAMM channel. Geometry of GAMM channel is shown in Fig. 4, where Γ_{uw} and Γ_{lw} are up-wall and down-wall, Γ_i is inlet of flow to channel and Γ_o is outlet of channel.

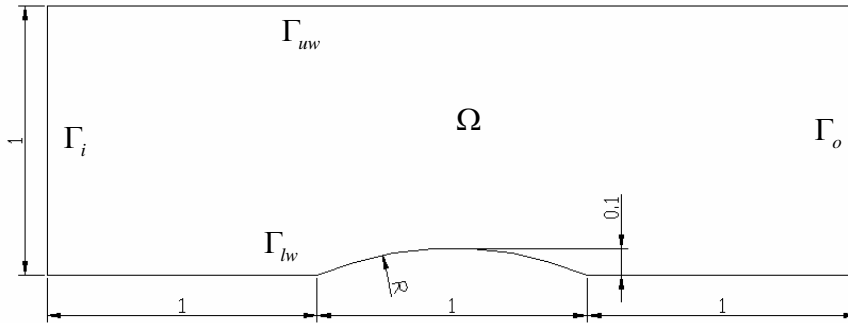


Fig. 4 Geometry of GAAM channel

We look for solution of inviscid, incompressible flows with artificial compressibility. This flow described of Euler equations (7). Constant pressure $\tilde{p} = 1$ and velocity $u = 1, v = 0$ was taken as the initial condition. Boundary conditions at the inlet are $u = 1$ and $v = 0$, \tilde{p} is extrapolated, at the outlet is $\tilde{p} = 1$ and u and v are extrapolated. At the walls we use impermeability condition ($\vec{u} \cdot \vec{n} = 0$). It is realized by the so-called reflexion. For numerical solution in 2D we created algebraically generated mesh with 75x25 cells.

Fig. 5 shows the distribution of pressure in GAMM channel in form of isolines. Computed distribution of pressure is not completely symmetric, but the difference from theoretically correct symmetric solution is small.

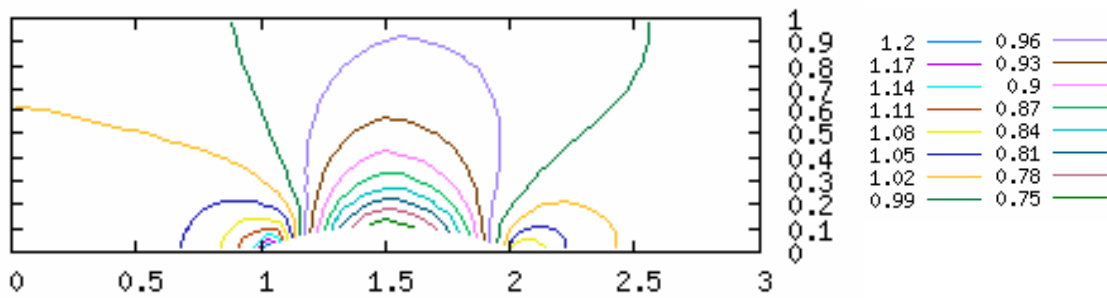


Fig. 5 Isolines of pressure in GAMM channel

4 CONCLUSIONS

We presented new numerical scheme (CE-SE) for one-dimensional and two-dimensional problems. In 1D we compared CE-SE scheme with upwind scheme and with exact solution. We detected oscillations generated by CE-SE scheme, nevertheless the results are better than those of upwind scheme. A software for solution of inviscid, incompressible flow in 2D channel has been developed. Results was satisfying. In future this scheme can be extended for compressible flows.

ACKNOWLEDGEMENTS

This work was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS 10/244/OHK2/3T/12

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