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# THE ANALYSIS OF DEFLECTION CURVES AT THE UNSYMMETRICAL BENDING

# ANALÝZA PRŮHYBOVÝCH KŘIVEK PŘI PROSTOROVÉM OHYBU

**Abstract**

The paper is devoted to the issue of accurate and approximate solution of deflection of beams at unsymmetrical bending. This article freely follows from the work [1], which demonstrated a possibility of solution of deflection curves with use of vector functions of scalar variable. It derived also accurate and approximate differential equation of deflection curve and compared the results of accurate and approximate solution of deflections using the example of simple fixed-end beam, which was however loaded by planar bending. This article deals with an example of unsymmetrical bending and it compares the results of accurate and approximate solutions not only of deflections, but also of size of radii of the first and second curvature at individual points of the fixed-end beam loaded by unsymmetrical bending.

**Abstrakt**

Příspěvek je věnován problematice přesného a přibližného řešení prohnutí nosníků při prostorovém ohybu. Článek volně navazuje na práci [1], ve které byla ukázána možnost řešení průhybových křivek pomocí vektorových funkcí skalárního argumentu. Byla v ní rovněž odvozena přesná a přibližná diferenciální rovnice průhybové křivky a provedeno srovnání výsledků přesného a přibližného řešení průhybů na příkladě jednoduchého vetknutého nosníku, namáhaného však ohybem rovinným. V tomto článku je řešen případ prostorového ohybu a jsou porovnány výsledky přesného a přibližného řešení nejen průhybů, ale i velikostí poloměrů první a druhé křivosti v jednotlivých bodech vetknutého nosníku, namáhaného prostorovým ohybem.

## 1 INTRODUCTION

Approximate differential equation instead of precise differential equation of deflection curve is usually used for solution of beam deflection. Derivation of both precise and approximate differential equations was demonstrated in the work [1] and an example of planar bending was used for comparison of results of calculated deflections made by both these approaches. This paper deals with an analysis of deflection curves at unsymmetrical bending.

## 2 THE SOLUTION OF DEFLECTION CURVE BY MEANS OF VECTOR FUNCTION OF SCALAR VARIABLE

Theory of vector function of scalar variable [2] was used for solution of the shapes of deflection curves in the work [1]. Scalar variable of vector function is usually length of the curve s, but it can be any parameter t.

$$\boldsymbol{a} = \boldsymbol{a}(s) \tag{1}$$

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At each point of the curve it is possible to determine three significant unit vectors $\boldsymbol{\tau}, \boldsymbol{n}$ and $\boldsymbol{b}$, which form main (or Frenet's ) original trihedron. These vectors are given by the relations:

$$\boldsymbol{\tau} = \frac{d\mathbf{a}}{ds}\ ;\ \mathbf{n} = \rho_1 \frac{d\boldsymbol{\tau}}{ds}\ ;\ \mathbf{b} = \boldsymbol{\tau} \times \mathbf{n} \tag{2}$$

And moreover this relation is valid:

$$\frac{d\mathbf{b}}{ds} = -\frac{\mathbf{n}}{\rho_2} \tag{3}$$

In these equations $\rho_1$ is radius of the first curvature in osculation plane and $\rho_2$ is radius of the second curvature in normal plane.

The following relation is valid for the first curvature $\underline{k}_1$ as reciprocal value of the radius of the first curvature $\rho_1$ :

$$k_1^2 = \frac{1}{\rho_1^2} = \frac{d\boldsymbol{\tau}}{ds} \bullet \frac{d\boldsymbol{\tau}}{ds} = \frac{d^2\mathbf{a}}{ds^2} \bullet \frac{d^2\mathbf{a}}{ds^2} = \frac{\|\ddot{\mathbf{a}}\|^2 \|\dot{\mathbf{a}}\|^2 - (\dot{\mathbf{a}} \bullet \ddot{\mathbf{a}})^2}{\|\dot{\mathbf{a}}\|^6} \tag{4}$$

The second curvature (torsion) is again as reciprocal value of the radius of the second curvature given by the relation:

$$k_2 = \frac{1}{\rho_2} = \frac{\dfrac{d\mathbf{a}}{ds} \bullet \left( \dfrac{d^2\mathbf{a}}{ds^2} \times \dfrac{d^3\mathbf{a}}{ds^3} \right)}{\dfrac{d^2\mathbf{a}}{ds^2} \bullet \dfrac{d^2\mathbf{a}}{ds^2}} \tag{5}$$

Derivation of these relations was shown in the work [1], this paper describes their use at analysis of unsymmetrical bending of beams.

## 3 UNSYMMETRICAL BENDING OF FIXED – END BEAM

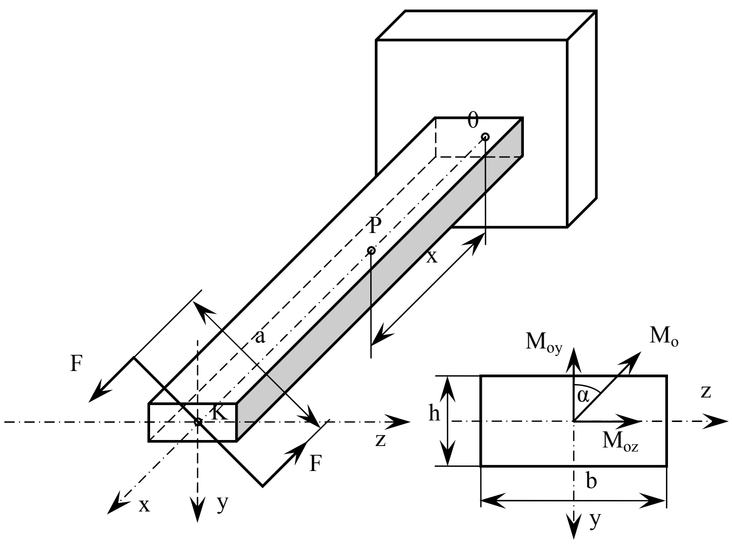


**Fig. 1** The unsymmetrical bending of the fixed – end beam

Analysis of deflection curve will be made on the example of fixed-end beam shown in Fig. 1.

The beam is at its loose end loaded with a bending moment, acting in a plane, which does not contain the main central axis of cross-section, which means that the beam is loaded by unsymmetrical bending. Solution uses with an advantage a principle of superposition, when the total bending moment $M_o = F \cdot a$ is divided into two components $M_{oy}$ and $M_{oz}$, which extract two planar deflections in mutually perpendicular axes $\underline{y}$ and $\underline{z}$. The points of the resulting deflection curve are then given by vector sums of deflections from both components of the bending moment $M_{oy}$ and $M_{oz}$.

### 3.1 The shape of deflection curve

#### 3.1.1 The accurate solution

The following relation was deduced in the work [1] for the case of the planar deflection for the first curvature of the deflection curve in the axis y:

$$k_{1y} = \frac{1}{\rho_{1y}} = \frac{y''}{\sqrt{[1+(y')^2]^3}} \tag{6}$$

Analogically for the first curvature of the deflection curve in the axis z the following will be valid:

$$k_{1z} = \frac{1}{\rho_{1z}} = \frac{z''}{\sqrt{[1+(z')^2]^3}} \tag{7}$$

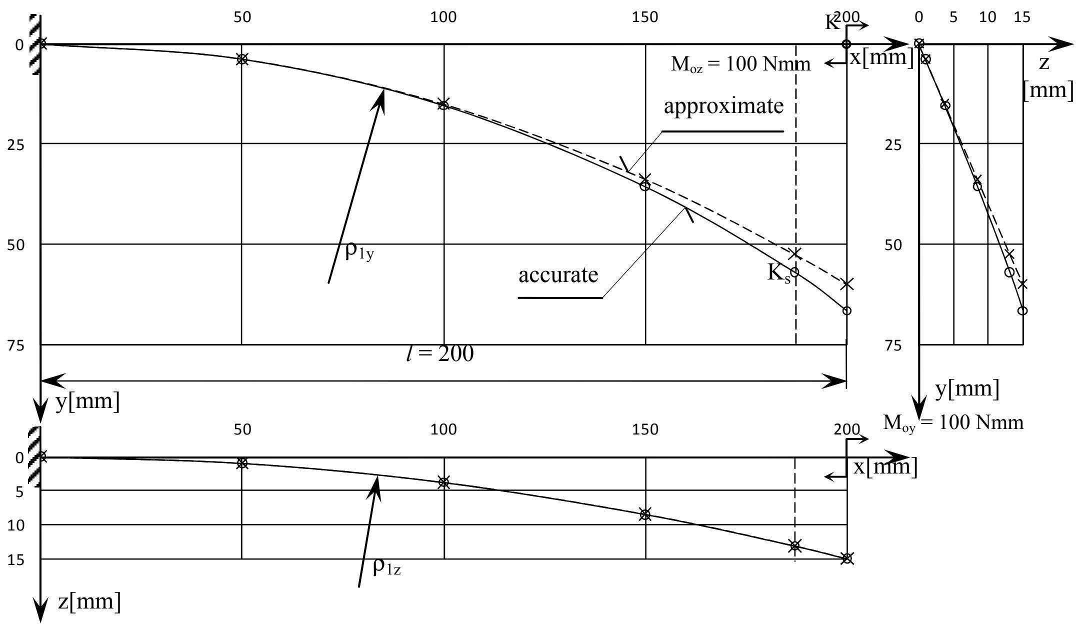


**Fig. 2** Deflection curves of the fixed – end beam

The following relation for the first curvature at the planar bending has been derived from the theory of elasticity:

$$k_1 = \frac{1}{\rho_1} = \frac{M(x)}{EI} \tag{8}$$

By comparing the right sides of the equations (6), (7) and (8) we obtain two differential equations for the components of deflections at unsymmetrical bending:

$$\frac{y_s''}{\sqrt{[1+(y_s')^2]^3}} = \pm\frac{M_{oz}}{EI_z}; \qquad \frac{z_s''}{\sqrt{[1+(z_s')^2]^3}} = \pm\frac{M_{oy}}{EI_y} \tag{9}$$

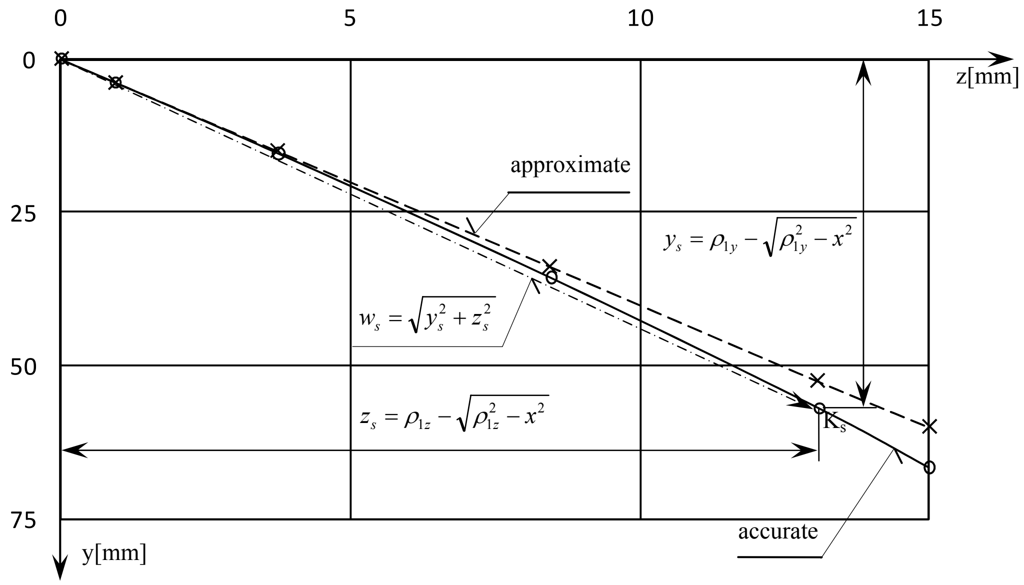


**Fig. 3** Look on deflection curves in direction of x – axis

Since the components of the bending moment $M_{oy}$ and $M_{oz}$ along the longitudinal axis of the beam x are constant in the solved example, the radii of the first curvature $k_{1y}$ and $k_{1z}$ are also constant and deflection curves in direction of the axes y and z are circles.

The solved beam and its deflection curves are plotted in Fig. 2. Longitudinal axis of not yet deformed beam matched with the coordinate axis x, and in front view, ground view and side view projections of spatial deflection curve into coordinate planes are visible. Fig. 3 shows side view in enlarged scale in direction of the axis z. Real spatial curve is plotted in figures in full continuous line. Dimensions and loads, as well as other calculated parameters of the beam are given in table 1, which contains also relations for their calculations.

Circles as projections of spatial deflection curve into coordinate planes have the following equations:

$$x^2 + (y_s - \rho_{1y})^2 = \rho_{1y}^2; \qquad x^2 + (z_s - \rho_{1z})^2 = \rho_{1z}^2 \tag{10}$$

The following is then valid for coordinates of spatial deflection curve:

$$x_s = x; \quad y_s = \rho_{1y} - \sqrt{\rho_{1y}^2 - x^2}; \qquad z_s = \rho_{1z} - \sqrt{\rho_{1z}^2 - x^2} \tag{11}$$

Resulting deflection of the loaded beam is then given by the vector sum of these coordinates:

$$w_s = \sqrt{y_s^2 + z_s^2} \tag{12}$$

**Tab. 1** Parameters of loaded fixed-end beam

| QUANTITY | | CALCULA-TION | DIMENSION | SIZE |
|---|---|---|---|---|
| $b$ | Width of beam cross – section | | mm | 2 |
| $h$ | Depth of beam cross – section | | mm | 1 |
| $l$ | Length of fixed – end beam | | mm | 200 |
| $E$ | Modulus of elasticity | | MPa | $2.10^5$ |
| $M_o$ | Bending moment | | Nmm | 141 |
| $\alpha$ | Angle of bending moment to y – axis | | ° | 45 |
| $M_{oy}$ | Bending moment to the y – axis | $M_o \cdot \cos\alpha$ | Nmm | 100 |
| $M_{oz}$ | Bending moment to the z – axis | $M_o \cdot \sin\alpha$ | Nmm | 100 |
| $W_{oy}$ | Bending section modulus to the y – axis | $\frac{h \cdot b^2}{6}$ | $mm^3$ | $0,\overline{6}$ |
| $W_{oz}$ | Bending section modulus to the z – axis | $\frac{b \cdot h^2}{6}$ | $mm^3$ | $0,\overline{3}$ |
| $I_y$ | Main axial quadratic moment of cross – section to y – axis | $\frac{h \cdot b^3}{12}$ | $mm^4$ | $0,\overline{6}$ |
| $I_z$ | Main axial quadratic moment of cross – section to z – axis | $\frac{b \cdot h^3}{12}$ | $mm^4$ | $0,1\overline{6}$ |
| $\sigma_{oy}$ | Bending stress from $M_{oy}$ | $\frac{M_{oy}}{W_{oy}}$ | MPa | 150 |
| $\sigma_{oz}$ | Bending stress from $M_{oz}$ | $\frac{M_{oz}}{W_{oz}}$ | MPa | 300 |
| $\sigma_{om}$ | Maximal bending stress | $\sigma_{oy} + \sigma_{oz}$ | MPa | 450 |
| $\rho_{1y}$ | Radius of the first curvature in x – y plane | $\frac{EI_z}{M_{oz}}$ | mm | $333,\overline{3}$ |
| $\rho_{1z}$ | Radius of the first curvature in x – z plane | $\frac{EI_y}{M_{oy}}$ | mm | $1333,\overline{3}$ |
| $\rho_1$ | Radius of the first curvature at the fixed point of beam | $\frac{\rho_{1y}\rho_{1z}}{\sqrt{\rho_{1y}^2 + \rho_{1z}^2}}$ | mm | 323,38 |
| $\rho_2$ | Radius of the torsion at the fixed point of beam | equation (29) | mm | $\infty$ |

Calculated deflections in several points of the solved beam are given in table 2.

**Tab. 2** The accuracy and approximate parameters of deflection curve.

| Parameter x | | mm | 50 | 100 | 150 | 200 | Calculation |
|---|---|---|---|---|---|---|---|
| *y* | accur. | mm | 3,77 | 15,35 | 35,66 | 66,66 | $y_s = \rho_{1y} - \sqrt{\rho_{1y}^2 - x^2}$ |
| | approx. | | 3,75 | 15 | 33,75 | 60 | $y_p = \frac{M_{oz}}{2EI_z} x^2$ |
| *z* | accur. | mm | 0,9378 | 3,7553 | 8,4643 | 15,085 | $z_s = \rho_{1z} - \sqrt{\rho_{1z}^2 - x^2}$ |
| | approx. | | 0,9375 | 3,75 | 8,4375 | 15 | $z_p = \frac{M_{oy}}{2EI_y} x^2$ |
| *w* | accur. | mm | 3,8862 | 15,8060 | 36,6478 | 69,9968 | $w_s = \sqrt{y_s^2 + z_s^2}$ |
| | approx. | | 3,8654 | 15,4615 | 34,7881 | 61,8466 | $w_p = \sqrt{y_p^2 + z_p^2}$ |
| *Δy* | | mm | 0,0213 | 0,3535 | 1,907 | 6,6666 | $\Delta y = y_s - y_p$ |
| | | % | 0,56 | 2,30 | 5,35 | 10 | $\Delta y = \frac{y_s - y_p}{y_s} \cdot 100$ |
| *Δz* | | mm | 0,0003 | 0,0053 | 0,0268 | 0,085 | $\Delta z = z_s - z_p$ |
| | | % | 0,032 | 0,14 | 0,32 | 0,56 | $\Delta z = \frac{z_s - z_p}{z_s} \cdot 100$ |
| *Δw* | | mm | 0,0208 | 0,3443 | 1,8578 | 8,1502 | $\Delta w = w_s - w_p$ |
| | | % | 0,53 | 2,18 | 5,07 | 11,64 | $\Delta w = \frac{w_s - w_p}{w_s} \cdot 100$ |
| *tg β* | accur. | | 0,2488 | 0,2446 | 0,2374 | 0,2263 | $tg\,\beta_s = \frac{z_s}{y_s}$ |
| | approx. | | 0,25 | 0,25 | 0,25 | 0,25 | $tg\,\beta_p = \frac{z_p}{y_p}$ |
| $\rho_1$ | accur. | mm | 324,64 | 328,13 | 332,03 | 337,32 | equation (20) |
| | approx. | | 335,04 | 370,90 | 433,20 | 525,70 | equation (25) |
| $\rho_2$ | accur. | mm | 3372 | 1711 | 1181 | 952 | equation (29) |
| | approx. | | ∞ | ∞ | ∞ | ∞ | equation (30) |

Load or shape of deflection curve of the solved beam doesn't depend on the beam length $l\left(x_K \cong 188{,}2mm\right)$. Derived relations for deflections but also for radii of the first and second curvatures are valid also for longer lengths of beams than the length given in this example.

### 3.1.2 Approximate solution

In the case of approximate solution the following approximate differential equations of deflection curve [1] are used instead of the equation (9):

$$y_p'' = \pm \frac{M_{oz}}{EI_z}; \qquad z_p'' = \pm \frac{M_{oy}}{EI_y} \tag{13}$$

Their solution results are coordinates of approximate deflection curve that are given by the following equations:

$$y_p = \pm \frac{M_{oz}}{2EI_z} x^2 = \frac{x^2}{2\rho_{1y}}; \quad z_p = \pm \frac{M_{oy}}{2EI_y} x^2 = \frac{x^2}{2\rho_{1z}} \tag{14}$$

It can be seen from the equations (14) that projections of approximate deflection curve into coordinate planes are parabolas. Similarly as in the case of accurate solution the resulting deflection was here also calculated according to the relation:

$$w_p = \sqrt{y_p^2 + z_p^2} \tag{15}$$

Results of calculations of approximate solution are also given in table 2. The table contains also the results of comparison of individual components and resulting deflections according to accurate and approximate solution. It can be seen that the resulting deflections of the solved beam differs even by more than 11%.

The fact that approximate deflection curve is planar curve, which is evidenced by the constant angle $\beta_p$, calculated from the following equation, is also very interesting:

$$tg\ \beta_p = \frac{z_p}{y_p} \tag{16}$$

Projections of approximate deflection curve are plotted in Figures 2 and 3 by dashed line.

## 3.2 The first curvature of deflection curve

### 3.2.1 Accurate solution

The equation (4) was used for calculation of radius of the first curvature. Length of the original not loaded beam x was chosen as scalar variable of vector function. Vector function and its derivations then have the following form:

$$\begin{aligned} &\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} && \dot{\boldsymbol{a}} \bullet \ddot{\boldsymbol{a}} = y'y'' + z'z'' \\ &\dot{\boldsymbol{a}} = \boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k} && \|\dot{\boldsymbol{a}}\| = \sqrt{1 + (y')^2 + (z')^2} \\ &\ddot{\boldsymbol{a}} = y''\boldsymbol{j} + z''\boldsymbol{k} && \|\ddot{\boldsymbol{a}}\| = \sqrt{(y'')^2 + (z'')^2} \end{aligned} \tag{17}$$

By substitution into the equation (4) we obtain:

$$k_{1s}^2 = \frac{1}{\rho_{1s}^2} = \frac{\left[(y'')^2 + (z'')^2\right]\left[1 + (y')^2 + (z')^2\right] - \left(y'y'' + z'z''\right)^2}{\left[1 + (y')^2 + (z')^2\right]^3} = \frac{(y'')^2 + \left(y'z'' - z'y''\right)^2 + (z'')^2}{\left[1 + (y')^2 + (z')^2\right]^3} \tag{18}$$

For deflection curve given by the equations (11) the following is then valid:

$$x_s = x\ ; \qquad y_s = \rho_{1y} - \sqrt{\rho_{1y}^2 - x^2} \qquad z_s = \rho_{1z} - \sqrt{\rho_{1z}^2 - x^2}$$

$$x_s' = 1\ ; \qquad y_s' = \frac{x}{\sqrt{\rho_{1y}^2 - x^2}} \qquad z_s' = \frac{x}{\sqrt{\rho_{1z}^2 - x^2}}$$

$$x_s'' = 0\ ; \qquad y_s'' = \frac{\rho_{1y}^2}{\sqrt{\left(\rho_{1y}^2 - x^2\right)^3}} \qquad z_s'' = \frac{\rho_{1z}^2}{\sqrt{\left(\rho_{1z}^2 - x^2\right)^3}} \tag{19}$$

$$x_s''' = 0\ ; \qquad y_s''' = \frac{3\rho_{1y}^2 x}{\sqrt{\left(\rho_{1y}^2 - x^2\right)^5}} \qquad z_s''' = \frac{3\rho_{1z}^2 x}{\sqrt{\left(\rho_{1z}^2 - x^2\right)^5}}$$

By substitution of these relations into the equation (18) we obtain:

$$k_{1s}^2 = \frac{1}{\rho_{1s}^2} = \frac{1}{\left(\rho_{1y}^2\,\rho_{1z}^2 - x^4\right)^3}\left\{\rho_{1y}^4\left(\rho_{1z}^2 - x^2\right)^3 + \left[x\rho_{1y}^2\left(\rho_{1z}^2 - x^2\right) - x\rho_{1z}^2\left(\rho_{1y}^2 - x^2\right)\right]^2 + \rho_z^4\left(\rho_{1y}^2 - x^2\right)^3\right\} \tag{20}$$

From this equation radii of the first curvature $\rho_{1s}$ in several points of the solved beam were then calculated, their values are given in table 2.

At the point of fixation (for $x = 0$) the following is then valid for radius of the first curvature $\rho_{1s}$:

$$\rho_{1s} = \frac{\rho_{1y} \cdot \rho_{1z}}{\sqrt{\rho_{1y}^2 + \rho_{1z}^2}} = 323{,}38\,mm \tag{21}$$

Literature gives for calculation of the first curvature also other formulas, which, however, lead to the same solution. For example the literature source [3] gives:

$$k_1 = \sqrt{\boldsymbol{\tau}' \bullet \boldsymbol{\tau}'} = \sqrt{\mathbf{a}'' \bullet \mathbf{a}''} = \sqrt{\left(x''\right)^2 + \left(y''\right)^2 + \left(z''\right)^2} \tag{22}$$

In this equation the length of the curve arc s is the scalar variable and derivations according to this variable are marked with an acute accent $(y')$.

When general variable t is used together with derivations according to this variable (for example x) then the following is valid for the first curvature (derivations are marked with a point):

$$k_1 = \frac{1}{(\dot{s})^2}\sqrt{(\ddot{x})^2 + (\ddot{y})^2 + (\ddot{z})^2 - (\ddot{s})^2} \tag{23}$$

### 3.2.2 Approximate solution

Equations of approximate deflection curve and their derivations are given by the relations:

$$\begin{array}{llll} x_p = x\ ; & x_p' = 1\ ; & x_p'' = 0\ ; & x_p''' = 0 \\ y_p = \dfrac{x^2}{2\rho_{1y}}\ ; & y_p' = \dfrac{x}{\rho_{1y}}\ ; & y_p'' = \dfrac{1}{\rho_{1y}}\ ; & y_p''' = 0 \\ z_p = \dfrac{x^2}{2\rho_{1z}}\ ; & z_p' = \dfrac{x}{\rho_{1z}}\ ; & z_p'' = \dfrac{1}{\rho_{1z}}\ ; & z_p''' = 0 \end{array} \tag{24}$$

Radius of the first curvature is calculated with use of the equation (18):

$$k_{1p}^2 = \frac{1}{\rho_{1p}^2} = \frac{(y'')^2 + (y'z'' - z'y'')^2 + (z'')^2}{\left[1 + (y')^2 + (z')^2\right]^3} = \frac{\rho_{1y}^4 \rho_{1z}^4 \left(\rho_{1y}^2 + \rho_{1z}^2\right)}{\left[\rho_{1y}^4 \rho_{1z}^4 + x^2\left(\rho_{1y}^2 + \rho_{1z}^2\right)\right]^3} \quad (25)$$

The calculated values for selected points of the beam are also given in table 2. At the point of fixation ( $x = 0$ ) this radius is equal to:

$$\rho_{1p} = \frac{\rho_{1y} \cdot \rho_{1z}}{\sqrt{\rho_{1y}^2 + \rho_{1z}^2}} = 323{,}38\, mm \quad (26)$$

It can be seen from equations (21) and (26) that at the point of fixation the radii of the first curvature of the accurate and approximate deflection curves are identical. However, at other points these radii differ.

### 3.3 The torsion of deflection curve

#### 3.3.1 Accurate solution

Calculation of radius of the second curvature can be made with use of the equation (5) or relations given in the literature source [3]:

$$k_2 = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{(x'')^2 + (y'')^2 + (z'')^2} ; \qquad k_2 = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{(\dot{s})^2\left[(\ddot{x})^2 + (\ddot{y})^2 + (\ddot{z})^2 - (\ddot{s})^2\right]} \quad (27)$$

The first expression in the equation (27) is valid, if the length of the curve s arc is the scalar variable of vector function $\boldsymbol{a}(s)$, the second expression is valid for general scalar variable. Mixed product of three vectors in the numerator of the equation (5) is expressed here in a form of determinant.

Radius of the second curvature will be calculated this time on the basis of the second relation of the equation (27) and length of non-deformed beam x will serve as scalar variable. Only the third derivation of coordinates of real deflection curve, which are given by the equation (19) and were already calculated, is needed for this calculation. The following partial calculations will be progressively made:

$$(\dot{s})^2 = 1 + (\dot{y})^2 + (\dot{z})^2 \; ; \qquad \dot{s} = \sqrt{1 + (\dot{y})^2 + (\dot{z})^2}$$

$$\ddot{s} = \frac{\dot{y}\,\ddot{y} + \dot{z}\,\ddot{z}}{\sqrt{1 + (\dot{y})^2 + (\dot{z})^2}} \; ; \qquad (\ddot{s})^2 = \frac{(\dot{y}\,\ddot{y} + \dot{z}\,\ddot{z})^2}{1 + (\dot{y})^2 + (\dot{z})^2}$$

$$(\ddot{x})^2 + (\ddot{y})^2 + (\ddot{z})^2 - (\ddot{s})^2 = (\ddot{y})^2 + (\ddot{z})^2 - \frac{(\dot{y}\,\ddot{y} + \dot{z}\,\ddot{z})^2}{1 + (\dot{y})^2 + (\dot{z})^2} = \frac{(\ddot{y})^2 + (\ddot{z})^2 + (\dot{y}\,\ddot{z} - \dot{z}\,\ddot{y})^2}{1 + (\dot{y})^2 + (\dot{z})^2}$$

$$(\dot{s})^2\left[(\ddot{x})^2 + (\ddot{y})^2 + (\ddot{z})^2 - (\ddot{s})^2\right] = (\ddot{y})^2 + (\ddot{z})^2 + (\dot{y}\,\ddot{z} - \dot{z}\,\ddot{y})^2$$

The following relation is then valid for torsion of deflection curve:

$$k_2 = \frac{\ddot{y}\,\dddot{z} - \ddot{z}\,\dddot{y}}{(\ddot{y})^2 + (\ddot{z})^2 + (\dot{y}\,\ddot{z} - \dot{z}\,\ddot{y})^2} \quad (28)$$

After substitution of derivations from the relation (19) into the equation (28) the following is valid for radius of the second curvature $\rho_{2s}$:

$$\rho_{2s} = \frac{\rho_{1y}^2\left(\rho_{1z}^2 - x^2\right) + \rho_{1z}^2\left(\rho_{1y}^2 - x^2\right) - 2x^2\left(\rho_{1y}^2 - x^2\right)\left(\rho_{1z}^2 - x^2\right)}{3x\left(\rho_{1y}^2 - \rho_{1z}^2\right)\sqrt{\left(\rho_{1y}^2 - x^2\right)\left(\rho_{1z}^2 - x^2\right)}} \tag{29}$$

Calculated radii of the second curvature $\rho_{2s}$ in selected points of the beam are given in table 2. It is obvious from the equation (29), that at the point of fixation (x = 0) this radius is infinitely big, and it is apparent from table 2 that with the increasing variable x it progressively decreases.

### 3.3.2 Approximate solution

After substitution of derivations of approximate deflection curve from the equation (24) into the equation (27) it is apparent that its numerator in the form of determinant is equal to zero:

$$\begin{vmatrix} 1 & \frac{x}{\rho_{1y}} & \frac{x}{\rho_{1z}} \\ 0 & \frac{1}{\rho_{1y}} & \frac{1}{\rho_{1z}} \\ 0 & 0 & 0 \end{vmatrix} = 0 \tag{30}$$

It means that torsion is in all points of approximate deflection curve zero and radius of the second curvature $\rho_{2p}$ is infinitely big, which means that this is a planar curve. The same conclusion was already drawn at plotting of the side view of the curve in Figures 2 and 3.

## 4 CONCLUSIONS

The authors made in their paper an attempt of mathematical analysis and geometrical interpretation of spatial deflection curve at load of the beam by unsymmetrical bending. Its shape was analysed and magnitudes of deflections were calculated, as well as radii of the first and second curvature at selected points of the beam. Vector function of scalar variable was used for solution and both accurate and approximate differential equations of deflection curve were derived with its help. Solution was made for the fixed-end beam subjected to load by unsymmetrical bending. The authors compared also magnitudes of deflections for accurate and approximate solution and they have determined differences of up to 11%. The basic textbooks of strength of materials [4], [5], [6] give the solutions of deflection curves by means of the approximate differential equation only. The interesting finding was also the fact that accurate solution result is spatial curve, while approximate solution gives a planar curve. The article may serve as suitable practical application at study of mathematics in the chapter on vector functions of scalar variable, as well as at study of theory of elasticity in the field of unsymmetrical bending.

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