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IMPLEMENTATION OF SHISHKIN MESH IN THE MODELLING OF SPRING-MASS SYSTEM

IMPLEMENTACE SHISKINOVA SÍTĚ V MODELOVÁNÍ HROMADNÉM PRUŽNÉM SYSTÉMU

Abstract

In this article we will analyze the model of spring-mass system with very large damping or very small mass as well as its numerical solution. In the first part of the work we will derive the model with initial conditions and discuss the behavior of the exact solution. In the next part we will analyze the numerical solution using the finite difference method and in the last part of the work we will implement the Shishkin mesh in the numerical scheme and comment the error of approximation.

Abstrakt

Tento článek se zabývá analýzou modelu hromadného pružného systému s velmi velkým tlumením nebo velmi malou hmotností, stejně jako její numerické řešení. V první části práce je odvozen model s počátečními podmínkami a diskuze o chování přesného řešení. V další části budeme analyzovat numerické řešení metodou konečných diferencí a v poslední části práce budeme provádět Shishkinovu síť v číselném schématu a komentovat chyby aproximace.

1 INTRODUCTION

We will analyze the spring that resist compression and extension. The spring also suspends extension, vertically from fixed support as shown in the following figure.

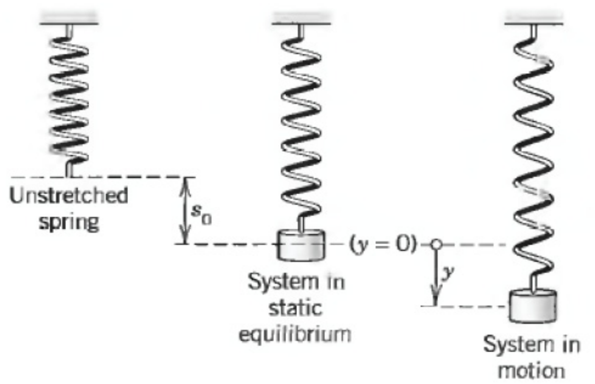

Fig. 1 Spring system

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At the lower end of the spring has been attached a mass *m*. We assume that the mass *m* is so large that we can neglect the mass of the spring.

Following the Newton's second law, we can conclude that the resultant of all the forces acting on the body is equal to *my''*, where *y(t)* is the displacement function.

According to Hooke's low the restoring force in the spring is equal to *–ky*, where *k* is the spring constant.

We will add to the system the damping force which is proportional to the velocity and is equal to *–cy'*, where *c* is the damping constant.

Now the observed system could be modeled by the following homogeneous differential equation:

$$my''+cy'+ky=0. \tag{1}$$

If we assume that the damping is very large or the mass is very small, i.e.

$$mk << c. \tag{2}$$

The equation (1) could be written in the form:

$$\varepsilon y''+y'+y=0, \tag{3}$$

where ε is small parameter.

We take the following initial conditions:

$$y(0)=A, \tag{4}$$

and

$$y'(0)=B. \tag{5}$$

In the following figure we can see the solution to the problem (3), (4), (5) for value 0,01 of small parameter.

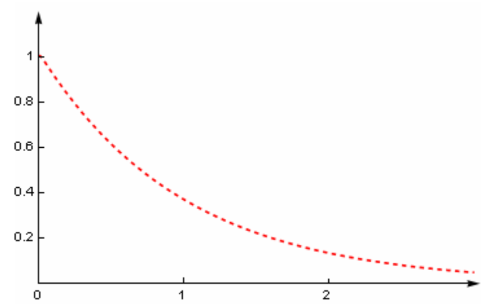

Fig. 2 The analytic solution for ε=0.01, A=B=1.

The function in the figure 2 is defined for any interval [0,t]. We can see that it rapidly decrease as time goes to infinity which tells us that we should distinguish two regions, one small region near zero and second region away from zero.

2 NUMERICAL TREATMENT OF THE PROBLEM

First we tried to use uniform mesh on the interval [0,3] using finite difference approximation of the equation (3). We used finite difference formulas as follows:

$$y'(t)\approx\frac{y_{i+1}^N-y_{i-1}^N}{2h}, \tag{6}$$

$$y''(t) \approx \frac{y_{i-1}^N - 2y_i^N + y_{i+1}^N}{h^2}, \tag{7}$$

where N is the number of points in the mesh, and h is defined by:

$$h = \frac{1}{N}. \tag{8}$$

Approximation formulas (6) and (7) produce the difference scheme with tridiagonal matrix as follows:

$$\begin{pmatrix} 0...0 & \frac{\varepsilon}{h^2} - \frac{1}{2h} & -\frac{2\varepsilon}{h^2} + 1 & \frac{\varepsilon}{h^2} + \frac{1}{2h} & 0...0 \end{pmatrix}. \tag{9}$$

Using the scheme described with matrix (8) we get the solution shown in figure 3.

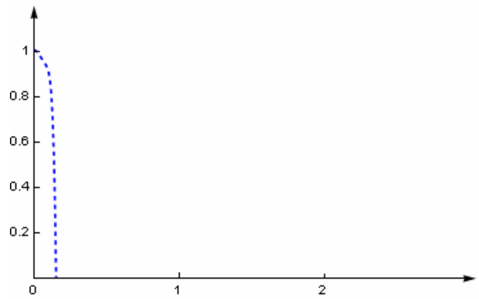

Fig. 3 The solution with finite differencing and uniform mesh.

As we can see in the figure 3 the solution blows up near zero. We can explain this using the theory of M-matrices. If we want to have stable numerical scheme it should have M-matrix, but the matrix (8) is not M-matrix when ε is small relative to h. To enforce this condition is impractical since it can lead to an intolerably large number of mesh points. To solve this problem we will use more tractable alternative which will be described in the next section.

3 SHISHKIN MESH

In the situation described above, where we have two regions for the solution, it seems reasonable to cluster mesh points where the solution is most troublesome instead of spreading them equidistantly over [0,1]. Since the early 1990s a simpler piecewise equidistant mesh has been propagated by Shishkin, Farell and Miller. Their idea is shown in the following picture.

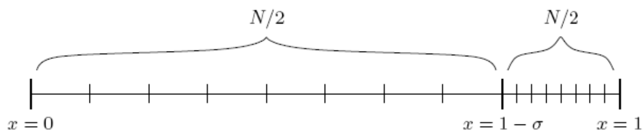

Fig. 4 The Shishkin mesh with transition point near 1.

In the figure 4, we can see the Shishkin type of mesh for the problem with boundary layer near 1. In our example we will use the transition point near 0 since our problem is defined for any interval [0,t].

As it was stated in [3] using the Shishkin mesh with appropriate choice of transition point leads to uniform convergent numerical scheme with the following bound:

$$\left| y_i - y_i^N \right| \le CN^{-1} \ln N, \tag{10}$$

where y_i is the value of analytic solution, and y_i^N is the numerical solution.

Some authors have been shown that the condition number of discrete linear system associated with this kind of problems is:

$$O\left(\varepsilon^{-2} N^2 \ln^{-2} N\right), \tag{11}$$

which is uncomfortably large when ε is small, but an easy preconditioning by diagonal scaling reduces the condition number.

4 NUMERICAL RESULT

In fig 5 we can see the improvement of the method (8) which is now used on Shishkin mesh.

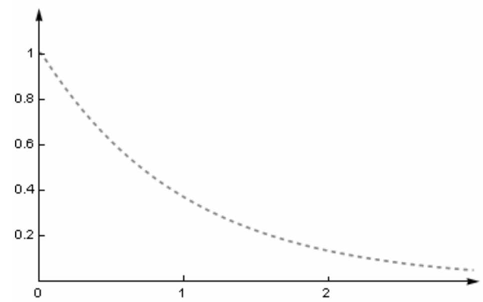

Fig. 5 The numerical results with Shishkin mesh with transition point near 0.

We used few different transition points near zero, but there is no significant influence to the quality of the numerical solution by changing the transition point. In the figure 5 we used 1 as transition point of the mesh.

5 CONCLUSIONS

In this work we tried to implement the Shishkin type of mesh which is originally made for convection-diffusion problems into spring-mass system with very large damping or very small mass which is singularly perturbated problem that form boundary layer near zero. We showed numerically that usage of Shishkin mesh can produce significantly better solution instead of using pure uniform mesh.

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