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A MATHEMATICAL MODEL OF ALLOCATION OF AIRCRAFT TO FLIGHTS OF REGULAR AIR ROUTES

MATEMATICKÝ MODEL ALOKACE LETADEL NA SPOJE PRAVIDELNÝCH LETECKÝCH LINEK

Abstract

The article deals with the problem of aircraft allocation to flights of air routes. For solving the problem, methods of linear programming that is a very universal solving tool [1] are employed. It contains the mathematical formulation of the problem, two linear mathematical models and a concrete example enabling the verification of functionality of mathematical models built. The solving of the concrete example was performed using optimization software Xpress-IVE.

Abstrakt

Článek se zabývá problémem nasazování letadel na spoje leteckých linek. K řešení problému využívá metod lineárního programování, které je velmi univerzálním řešícím nástrojem [1]. Obsahuje matematickou formulaci problému, dva lineární matematické modely a konkrétní příklad umožňující ověření funkčnosti sestavených matematických modelů. Řešení konkrétního příkladu proběhlo v optimalizačním software Xpress-IVE.

1 INTRODUCTION

At present, the management of air routes is usually solved in the following way. In the territory of every state, one main airport is built by which the given state is connected with important airports of other states. The air connection of other airports of the given state with this main airport is ensured by domestic air routes. The number of flights of domestic air routes depends on demand for transport during the day; most frequently, the connection of main airport with domestic airports is realized, however, by means of one pair of flights as a minimum. During evening hours, aircraft depart from the main airport for domestic airports, during morning hours aircraft depart in the opposite direction. What is characteristic of the mentioned situations is the fact that the aircraft in its final destination remains parking during night hours.

Nevertheless, the mentioned situation does not apply merely to domestic air routes; we face a similar phenomenon even in the case of a number of international air routes.

2 PROBLEM FORMULATION

We have the following problem, which is based on a simple task [2]. The air carrier is to decide, for the following period (during the drawing-up of flight schedule), the allocation of specific types of aircraft to flights to the planned number of destinations operated in a certain time period.

Furthermore, let us consider for simplification that for each of destinations considered, merely one flight leaves on a certain day in evening hours and that from each of the destinations considered,

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one flight (pair of flights will be designated in abbreviation as destination and for the other destinations we would proceed analogically) leaves in morning hours of the following day in the opposite direction as well. Let us consider that the air routes to destinations will be realized without servicing other intermediate airports. Moreover, let us consider that aircraft of each of the types considered are able to reach each of destinations without en-route stops for technical reasons (e.g. aircraft refuelling). The carrier knows the intensities of passengers of individual destinations (correspond to e.g. the upper quartile of statistical investigation – determination of the number of passengers), also knows the numbers of aircraft according to individual types that can be allocated to the service of destinations. As well, the carrier must respect a certain reserve of aircraft intended for the covering of possible failures of aircraft of the same type (in the framework of this amount, the aircraft that must be immobilized due to regular technical inspections can be considered). The requirement of the carrier is to find such a solution that would minimize the total costs of service of defined destinations in the period being dealt with.

For solving the given problem, linear programming will be used. The reason is that this is a quite universal solving tool; for solving linear models a large number of solving computer programs exist and tasks can be even considerably extensive [3].

Now, we shall select individual variables that will represent individual decisions in the model and we shall determine their domains of definition. In our case we shall use a bivalent variable. Its values will determine whether or not the type of aircraft will be allocated to the destination. Let us designate the chosen bivalent variable i.e. by the symbol x_{ij}. If the variable x_{ij} acquires a value of 1, it will mean that the aircraft of the type j will be allocated to the service of destination i. If the variable x_{ij} acquires a value of 0, it will mean that the aircraft of the type j will not be allocated to the service of destination i.

3 MATHEMATICAL MODEL

At first we shall recapitulate requirements for the optimization criterion and the input value list. In the course of optimization, we minimize the total costs of servicing all destinations. We shall designate:

I .... set of destinations,

J ... set of aircraft types,

N_j . set of aircraft of the type j [aircraft],

z_j ... number of reserve aircraft of the type j [aircraft],

n_j ... costs of one flying hour of the aircraft of the type j [pj.h^{-1}] (let us consider costs related to the ideal flight level)

t_{ij} ... flying time of the aircraft of the type j into the destination i [h] (different flying times into the same destination are assumed for the type of used aircraft and the flying times into the destination i are the same in both directions for the aircraft of the type j),

$\overline{n_{ij}}$ .. costs of ground time of the aircraft of the type j for 1 hour at the airport of the destination i [pj.h^{-1}],

$\overline{t_{ij}}$ ... ground time of the aircraft of the type j at the airport of the destination i [h],

q_i ... intensity of passengers of the flight i [passengers.flight^{-1}] (intensity of passengers per flight is considered, which is higher, thereinafter so-called decisive passenger intensity),

k_j ... capacity of the aircraft of the type j [seats].

The linear mathematical model of the task has the following shape:

$$\sum_{i \in I} \sum_{j \in J} \left(2t_{ij} n_j + \overline{t_{ij} n_{ij}} \right) x_{ij} \to \min \tag{1}$$

on the following conditions

$$\sum_{i \in I} x_{ij} \le N_j - z_j \text{ for } j \in J \tag{2}$$

$$\sum_{j \in J} k_j x_{ij} \ge q_i \text{ for } i \in I \tag{3}$$

$$\sum_{j \in J} x_{ij} = 1 \text{ for } i \in I \tag{4}$$

$$x_{ij} \in \{0;1\} \text{ for } i \in I \quad j \in J \tag{5}$$

The expression (1) represents an objective function. The conditions (2) will ensure that for individual types of aircraft, more aircraft than are available for the carrier will not be allocated to destinations. The conditions also take into account the maintenance of required reserve of individual types of aircraft. The conditions (3) will ensure that to routes such aircraft will be allocated that will, from the point of view of capacity, comply with the decisive intensities of passengers of individual destinations. The conditions (4) will ensure that to every destination, just one aircraft will be allocated. The conditions of the type (5) are obligatory conditions.

However, there is also another approach to solving the problem. The author of this idea is Prof. RNDr. Jaroslav Janáček, CSc. At defined simplifications we are able to transform the task being solved into the transport task, in which the capacities of sources and the requirements of consumers are integers. The sources will be represented by individual types of aircraft; the consumers will be represented by individual destinations. The capacities of sources will be equal to the numbers of aircraft in individual groups; the requirements of consumers will be equal to 1. In the course of solving we shall utilize the basic sequence of linear programming. This sentence is proved, e.g. in [3] and says: if the capacities of sources and the requirements of consumers are integers, each basic feasible solution of the task of linear programming will be an integer. After this transformation it is not necessary to put bivalent variables into the task. However, variables must merely be introduced for permissible relations (in situations when the aircraft can be assigned to the destination).

To be able to realize the proposal by Prof. Janáček and to write the model in a symbolic form, we shall introduce two new sets I_j and J_i. The set I_j will represent the destinations, to which we can allocate the aircraft of the type j. The set J_i will represent types of aircraft able to reach the destination i. The aircraft of the type j can be allocated to the destination i only in the event that $k_j \ge q_i$.

The linear mathematical model will have the form given below:

$$\sum_{i \in I_j} \sum_{j \in J_i} \left(2t_{ij}n_j + \overline{t_{ij}n_{ij}}\right)x_{ij} \rightarrow \min \quad (6)$$

on the following conditions

$$\sum_{i \in I_j} x_{ij} \leq N_j - z_j \text{ for } j \in J \quad (7)$$

$$\sum_{j \in J_i} x_{ij} = 1 \text{ for } i \in I \quad (8)$$

$$x_{ij} \geq 0 \text{ for } i \in I_j \quad j \in J_i \quad (9)$$

The expression (6) is an objective function. The conditions (7) will ensure that not more aircraft than are available for the carrier will be allocated to the destinations. The conditions (7) also consider here the maintenance of required reserve of individual aircraft types. The conditions (8) will ensure that to each destination, just one aircraft will be allocated. The conditions of the type (9) are obligatory conditions.

The advantage of the latter model is the replacement of bivalent variables by nonnegative variables. This replacement will make it possible to solve more extensive tasks, and in the case of extensive tasks then also to shorten the calculation time.

Note to the objective function: it is not necessary to include the coefficient 2 in the objective function of both the models. If we put it into the objective function, we shall obtain the actual amount of costs after optimization (if the flying times in both the directions are equal). If we did not put it there the optimal solution would also be achieved, but the actual amount of costs would have to be calculated.

4 EXPERIMENTAL PART

Now let us demonstrate the functionality of the built mathematical model by means of a concrete example. The air carrier has planned 8 destinations and is to decide the allocation of 5 types of aircraft. The numbers of aircraft by type, which are available for the air carrier in the given period, are presented in Table No. 1.

Tab. No. 1

Aircraft type	Aircraft capacity [seats]	Total number of aircraft	Obligatory reserve of aircraft
1	42	4	2
2	72	3	1
3	34	6	2
4	118	4	2
5	190	3	1

Decisive intensities of passengers of individual relations in the given period and additional information required for the setup of the mathematical model of the proposed type are given in Tab. No. 2.

Tab. No. 2

Destination number	Decisive intensity of passengers [pass.aircraft⁻¹]	Costs of flying hour [thousand CZK.h⁻¹]		Flying time in one direction [h]		Costs of 1 hour of ground time [thousand CZK.h⁻¹]		Ground time [h]	
		aircraft type	costs	aircraft type	time	aircraft type	costs	aircraft type	time
		1	120	1	3	1	0.104	1	6
		2	150	2	2.85	2	0.128	2	6.3
1	58	3	180	3	2.7	3	0.08	3	6.6
		4	250	4	1.65	4	0.52	4	8.7
		5	270	5	1.5	5	0.56	5	9
		1	120	1	2.5	1	0.13	1	9
		2	150	2	2.375	2	0.16	2	9.25
2	33	3	180	3	2.25	3	0.1	3	9.5
		4	250	4	1.375	4	0.65	4	11.25
		5	270	5	1.25	5	0.7	5	11.5
		1	120	1	1.5	1	0.091	1	12
		2	150	2	1.425	2	0.112	2	12.15
3	29	3	180	3	1.35	3	0.07	3	12.3
		4	250	4	0.825	4	0.455	4	13.35
		5	270	5	0.75	5	0.49	5	13.5
		1	120	1	1	1	0.117	1	10
		2	150	2	0.95	2	0.144	2	10.1
4	74	3	180	3	0.9	3	0.09	3	10.2
		4	250	4	0.55	4	0.585	4	10.9
		5	270	5	0.5	5	0.63	5	11
		1	120	1	1.5	1	0.091	1	10
		2	150	2	1.425	2	0.112	2	10.15
5	33	3	180	3	1.35	3	0.07	3	10.3
		4	250	4	0.825	4	0.455	4	11.35
		5	270	5	0.75	5	0.49	5	11.5
		1	120	1	2	1	0.104	1	10
		2	150	2	1.9	2	0.128	2	10.2
6	50	3	180	3	1.8	3	0.08	3	10.4
		4	250	4	1.1	4	0.52	4	11.8
		5	270	5	1	5	0.56	5	12
		1	120	1	2.5	1	0.143	1	8
		2	150	2	2.375	2	0.176	2	8.25
7	66	3	180	3	2.25	3	0.11	3	8.5
		4	250	4	1.375	4	0.715	4	10.25
		5	270	5	1.25	5	0.77	5	10.5
		1	120	1	3.5	1	0.13	1	8
		2	150	2	3.325	2	0.16	2	8.35
8	63	3	180	3	3.15	3	0.1	3	8.7
		4	250	4	1.925	4	0.65	4	11.15
		5	270	5	1.75	5	0.7	5	11.5

Note: all costs incurred abroad are converted into CZK.

We do not present concrete variants of mathematical models owing to their extensiveness. In the first mathematical model the objective function (1) has 40 members; the set of limiting conditions contains 61 limiting conditions. The number of conditions (2) is 5, the number of conditions (3) is 8, the number of conditions (4) is 8 and the number of conditions (5) is 40.

In the other mathematical model, the number of members of the objective function (6) is equal to the number of variables established. The number of conditions (7) is 5, the number of conditions (8) is 8, and the number of conditions (9) is equal to the number of variables set.

The optimization calculation was performed using optimization software Xpress-IVE for both the models. After the calculation, the value of the objective function at optimal solution was CZK 4 827. 46 thousand. After the solution of both the models, the optimal allocations of aircraft to individual destinations were the same, see Tab. No. 3.

Tab. No. 3

Destination number	Decisive intensity of passengers	Aircraft type	Aircraft capacity	Destination number	Decisive intensity of passengers	Aircraft type	Aircraft capacity
1	58	5	190	5	110	4	118
2	62	2	72	6	50	2	72
3	29	1	42	7	66	2	72
4	74	4	118	8	63	5	190

We want to call attention to one fact that shows the advantage of use of linear programming. It is the destination No. 4 and the destination No. 8 that have the highest decisive intensities of passengers, i.e. 74 and 63, respectively. Thus it would be logical to allocate aircraft of highest capacity to these destinations. However, in the case of destination No. 4, which has even the highest decisive intensity of passengers, this did not happen. The aircraft of highest capacity (190 seats) was, on the contrary, allocated to the destination No. 1 (58 passengers). Both the solutions are permissible, the allocation of aircraft of the capacity of 190 seats to the destination No. 4 seems to be more logical. From the point of view of optimization criterion, this does not however hold true.

Nevertheless, we must realize that the decisive intensities of passengers are not a single input parameter, because the costs of 1 hour of flying time and of ground time of aircraft at airports of final destinations are key parameters. And this is what decides. Let us make sure of it.

Let us assume that in the case of destinations Nos. 2, 3 and 5 to 8, the allocated type of aircraft does not change.

According to the linear model we have:
the aircraft with the capacity of 190 seats (type 5) which we shall allocate to the destination No. 1 – costs will be CZK 815.04 thousand, the aircraft with the capacity of 118 seats (type 4) which we shall allocate to the destination No. 4 – costs will be CZK 281.3765 thousand.

In the case of intuitive allocation of aircraft we have:
the aircraft with the capacity 190 seats (type 5) which we shall allocate to the destination No. 4 – costs will be CZK 276.93 thousand, the aircraft with the capacity of 118 seats (type 4) which we shall allocate to the destination No. 1 – costs will be CZK 829.524 thousand.

At the intuitive manner of aircraft allocation we shall acquire the value of objective function by CZK 10. 0375 thousand higher (worse solution) – the solution would not be optimal at this manner of allocation. Naturally, the saved costs are not high, but we must realize that we could make similar errors in more extensive tasks more than one at the intuitive manner of solving. In addition, the costs of CZK 10. 0375 thousand would be paid uselessly repeatedly (if we fly the given relation daily, we would save the amount of CZK 3 663.6875 thousand annually).

When we use linear programming for solving, we have certainty of achieving the optimal solution.

5 CONCLUSION

In the submitted article we present one of possible applications of linear programming devoted to the allocation of aircraft to the number of destinations known in advance. The model can be used in the design of carrier's flight schedule. The total costs of servicing all destinations being dealt with are the optimization criterion. The article contains the formulation of the problem, mathematical models and calculation experiment. The experiment was concerned with the task of the extent of 5 aircraft types and 8 destinations. Calculations executed in optimization software Xpress-IVE have verified the functionality of the models; the time of calculation for the given extent of task did not exceed 0.1 s in any case.

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