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Pavel MACURA*

PRECISION CHECK OF BEAM BENDING SOLUTION BY MEANS OF ANALYTICAL METHOD

POSOUZENÍ PŘESNOSTI ŘEŠENÍ PROHNUTÍ NOSNÍKŮ ANALYTICKOU METODOU

Abstract

The paper is devoted to the issues of beam bending with focus on one of possible methods of solution – by analytical method. Principle of this method is based on solution of approximate differential equation of the bending curve, derived from the precise differential equation of the bending curve with omission of one of its member. The bending curves can be spatial or in some simpler cases planar, and it is appropriate to use for their solution a vector analysis, specifically vector functions of the scalar argument. With use of them at first the precise and approximate differential equation of the bending curve is determined and then the deviations between the accurate and approximate solutions are assessed.

Abstrakt

Příspěvek je věnován problematice řešení prohnutí nosníků se zaměřením na jednu z možných metod řešení, a to řešení pomocí analytické metody. Princip metody je založen na řešení přibližné diferenciální rovnice průhybové křivky, odvozené z přesné diferenciální rovnice průhybové křivky zanedbáním jednoho jejího členu. Průhybové křivky mohou být prostorové nebo i v jednodušších případech rovinné a pro jejich řešení je vhodné použít vektorovou analýzu, konkrétně vektorové funkce skalárního argumentu. Pomocí nich je zde nejdříve určena přesná i přibližná diferenciální rovnice průhybové křivky a následně posouzeny odchylky přesného a přibližného řešení.

1 INTRODUCTION

Numerous calculation methods are available for solution of the beam deflection, apart from analytical method it is possible to use also e.g. graphical-analytical method, graphical a Castigliano's methods. Result of analytical method is an equation of the bending curve and deflection at any point of the beam deflection can be calculated by simple insertion of the coordinates of that point into the equation of the bending curve. We will indicate below first the mathematical procedure for determination of differential equation of the bending curve and then accurate and approximate analytical solution on the example of a simple beam.

2 DIFFERENTIAL EQUATION OF BENDING CURVE

Vector function of the scalar variable has been used for determination of differential equation of the bending curve, which is part of the vector analysis. That's why we first derive and define some important notions and quantities, used in a vector analysis at solution of planar and spatial curves [1].

2.1 Bending curve as vector function of scalar variable

A beam as a flexible body changes as a result of load its shape, which is in practice manifested by creation of the bending curve. Fig. 1 shows as an example a bending of the initially straight simply supported beam.

* prof. Ing. DrSc., Department of Mechanics of Materials, Faculty of Mechanical Engineering, VŠB – TU Ostrava, 17. listopadu 15, Ostrava – Poruba, tel. (+420) 59 732 3598, e-mail pavel.macura@vsb.cz

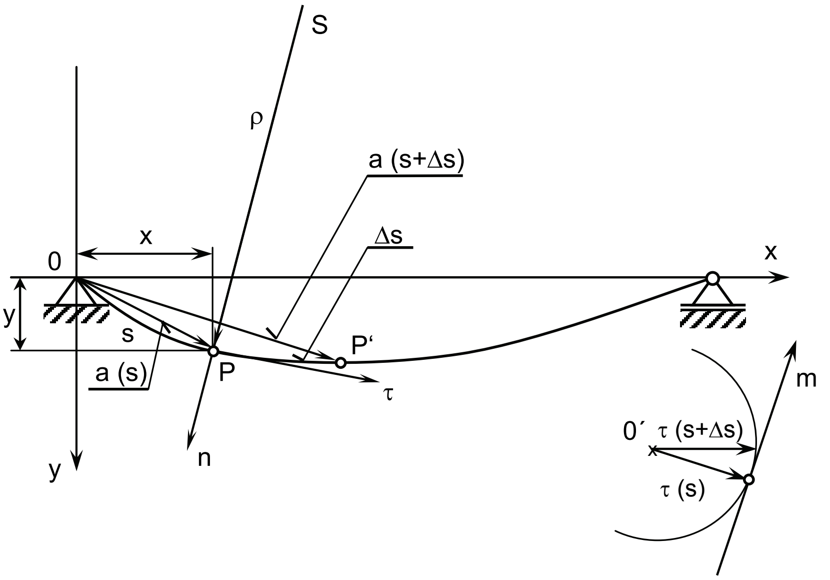

Fig. 1 Defection curve as a vector function of scalar variable

Position of each point P on the bending curve can be characterised by the vector $\boldsymbol{a}(s)$, connecting this point with the coordinate basic origin O. Position of the point P on the bending curve can be characterised by the length of the arc of the bending curve s between the points O and P. The vectors $\boldsymbol{a}(s)$ then form the vector function $\boldsymbol{a}(s)$ of the scalar variable s:

$$\boldsymbol{a} = \boldsymbol{a}(s) \tag{1}$$

End points of the vectors $\boldsymbol{a}(s)$, which start at the point O, fill the curve called hodograph of vector function of the scalar variable. In this case the beam bending curve is the hodograph of the vector function $\boldsymbol{a}(s)$.

Derivation of the vector function of the scalar variable is of great importance in the vector analysis. It is defined similarly as in differential calculus:

$$\dot{\boldsymbol{a}}(s) = \frac{d\boldsymbol{a}(s)}{ds} = \lim_{\Delta s \to 0} \frac{\boldsymbol{a}(s+\Delta s) - \boldsymbol{a}(s)}{\Delta s} = \boldsymbol{\tau} \tag{2}$$

Geometric meaning of derivation of the vector function of the scalar variable is obvious also from Fig. 1. The vector, connecting the points P and P‘ is a difference of vectors $\boldsymbol{a}(s+\Delta s)$ and $\boldsymbol{a}(s)$, and limit in the equation (2) is formed by the unit vector $\boldsymbol{\tau}$, which has direction of a tangent to a hodograph of the function $\boldsymbol{a}$(s) at the point P, in this particular case direction of a tangent to the bending curve.

Derivation of the above mentioned vector function $\boldsymbol{\tau}(s)$ according to the variable s plays another important role in the vector analysis:

$$\frac{d\boldsymbol{\tau}}{ds} = \frac{d^2\boldsymbol{a}}{ds^2} = \boldsymbol{m} = \frac{\boldsymbol{n}}{\rho} = k\,\boldsymbol{n} \tag{3}$$

Hodograph of the vector function $\boldsymbol{\tau}(s)$ of the scalar variable s is a circle of radius equal to one, which is also shown in Fig. 1. Result of derivation according to the equation (3) is the vector $\boldsymbol{m}$, which is perpendicular to the vector $\boldsymbol{\tau}$. However, it is not unit vector, since the parameter s is not the length of the arc of hodograph of the vector function $\boldsymbol{\tau}(s)$, but of the beam bending curve. The vector $\boldsymbol{m}$ is called the vector of the first (reflex) curvature of the bending curve, and the plane, determined by the vectors $\boldsymbol{\tau}$ and $\boldsymbol{n}$, is so called osculating plane of the bending curve at the point P. Normal line to the bending curve, which is perpendicular to a tangent of the bending curve at the point P and which is lying in the osculating plane, is called the main normal line of the bending curve. If ***n*** is the unit vector of the main normal line in direction of the vector ***m,*** then the quantity ρ in the equation (3) is a radius of the first curvature of the bending curve and its inverse value *k* is so called first curvature. The point *S* in Fig. 1, for which the following equation is valid

$$\vec{PS} = \rho \boldsymbol{n} \tag{4}$$

is called the centre of the first curvature (flexion) of the bending curve at the point P.

The equation (3) can be used for deriving of a procedure for calculation of radius of the first curvature of the bending curve. The following is valid for the unit vector ***n***:

$$\boldsymbol{n} = \rho \frac{d\tau}{ds} \tag{5}$$

The equation (5) can be modified to the form:

$$\boldsymbol{1} = \rho^2 \frac{d\tau}{ds} \cdot \frac{d\tau}{ds} = \rho^2 \frac{d^2\boldsymbol{a}}{ds^2} \cdot \frac{d^2\boldsymbol{a}}{ds^2} \tag{6}$$

The following expression is then valid for the first curvature of the bending curve:

$$k^2 = \frac{1}{\rho^2} = \frac{d\tau}{ds} \cdot \frac{d\tau}{ds} = \frac{d^2\boldsymbol{a}}{ds^2} \cdot \frac{d^2\boldsymbol{a}}{ds^2} \tag{7}$$

The unit tangent vector to the bending curve **τ** can be also described as:

$$\tau = \frac{\dot{\boldsymbol{a}}}{\|\dot{\boldsymbol{a}}\|} \tag{8}$$

where $\|\dot{\boldsymbol{a}}\|$ is the norm of the vector $\dot{\boldsymbol{a}}$.

After derivation and modification of the equation (8) we get:

$$\frac{d\tau}{ds} = \frac{d}{ds}\left(\frac{\dot{\boldsymbol{a}}}{\|\dot{\boldsymbol{a}}\|}\right) = \frac{\ddot{\boldsymbol{a}}}{\|\dot{\boldsymbol{a}}\|^2} - \frac{\dot{\boldsymbol{a}}}{\|\dot{\boldsymbol{a}}\|^4}(\dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}})^2 \tag{9}$$

The following expression is then valid for the first curvature of the bending curve:

$$k^2 = \frac{1}{\rho^2} = \frac{d\tau}{ds} \cdot \frac{d\tau}{ds} = \frac{\|\ddot{\boldsymbol{a}}\|^2 \|\dot{\boldsymbol{a}}\|^2 - (\dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}})^2}{\|\dot{\boldsymbol{a}}\|^6} \tag{10}$$

For solution of beams and their bending curves it is more appropriate to use the vector function ***a*** not as a function of the scalar variable *s*, but as a function of the variable *x*. In that case it is possible to describe the vector function **τ** in this form:

$$\boldsymbol{\tau} = \frac{d\boldsymbol{a}}{ds} = \frac{d\boldsymbol{a}}{dx}\frac{dx}{ds} \tag{11}$$

The following relations are then valid for the vector function ***a*** for the new scalar variable *x* according to the Fig. 1:

$$\left.\begin{array}{ll} \boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j} & \|\dot{\boldsymbol{a}}\| = \sqrt{1+(y')^2} \\ \dot{\boldsymbol{a}} = \boldsymbol{i} + y'\boldsymbol{j} & \|\ddot{\boldsymbol{a}}\| = y'' \\ \ddot{\boldsymbol{a}} = y''\boldsymbol{j} & \dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}} = (\boldsymbol{i} + y'\boldsymbol{j}) \cdot y''\boldsymbol{j} = y'y'' \end{array}\right\} \tag{12}$$

The equation (10) for the first curvature of the planar bending curve can be then modified to the form:

$$k^2 = \frac{1}{\rho^2} = \frac{[1+(y')^2](y'')^2 - (y'y'')^2}{[1+(y')^2]^3} = \frac{(y'')^2}{[1+(y')^2]^3} \tag{13}$$

In case of spatial bending and creation of spatial bending curve it is necessary to define for its analysis and calculation also other quantities of vector function of the scalar variable. This concerns in the first place the unit vector ***b*** at the point P, for which the following relation is valid:

$$\boldsymbol{b} = \tau \times \boldsymbol{n} \tag{14}$$

It is evident from the equation (14), that the vector ***b*** is perpendicular to the vectors τ and ***n***, and it is therefore perpendicular also to the osculation plane of the bending curve at the point P. It is called unit vector of binormal to the bending curve at this point.

At solution of spatial bending curves the derivation of the vector function ***b***(*s*) according to the variable *s* is also of great importance. The following equation is valid for it:

$$\frac{d\boldsymbol{b}}{ds} = \frac{d(\tau \times \boldsymbol{n})}{ds} = \frac{d\tau}{ds} \times \boldsymbol{n} + \tau \times \frac{d\boldsymbol{n}}{ds} = \tau \times \frac{d\boldsymbol{n}}{ds} \tag{15}$$

Result of this derivation is a vector, which is perpendicular to the vectors τ and ***b***, and which lies at the main normal line to the bending curve. It is not a unit and the following relation is valid for its magnitude:

$$\frac{d\boldsymbol{b}}{ds} = -\frac{\boldsymbol{n}}{T} = \kappa\boldsymbol{n} \tag{16}$$

The value *T* is here called a radius of the second curvature of the bending curve and its inverse value κ is called its torsion.

The unit vectors τ, ***n*** and ***b*** form at each point of the bending curve a rectangular and dextrorotatory system, called main or also the Frenet's trihedron. Apart from the already mentioned osculation plane at the point of the bending curve a normal plane is defined, which is determined by the vectors ***n*** and ***b***, and rectification plane, determined by the vectors τ and ***b***.

The following relation is valid for the magnitude of torsion κ from the equation (16):

$$\kappa = \frac{1}{T} = -\frac{d\boldsymbol{b}}{ds} \cdot \boldsymbol{n} \tag{17}$$

Torsion κ can be expressed with use of the equation (14) also in the form:

$$\kappa = \frac{1}{T} = \tau \cdot \left(\boldsymbol{n} \times \frac{d\boldsymbol{n}}{ds} \right) \tag{18}$$

These relations are valid for the vector functions in the equation (18):

$$\tau = \frac{d\boldsymbol{a}}{ds} \;\; ; \;\; \boldsymbol{n} = \frac{d^2\boldsymbol{a}}{ds^2}\rho \;\; ; \;\; \frac{d\boldsymbol{n}}{ds} = \frac{d^3\boldsymbol{a}}{ds^3}\rho \tag{19}$$

Torsion κ of the spatial bending curve is then given by the relation:

$$\kappa = \frac{1}{T} = \rho^2 \frac{d\boldsymbol{a}}{ds} \cdot \left(\frac{d^2\boldsymbol{a}}{ds^2} \times \frac{d^3\boldsymbol{a}}{ds^3} \right) = \frac{\frac{d\boldsymbol{a}}{ds} \cdot \left(\frac{d^2\boldsymbol{a}}{ds^2} \times \frac{d^3\boldsymbol{a}}{ds^3} \right)}{\frac{d^2\boldsymbol{a}}{ds^2} \cdot \frac{d^2\boldsymbol{a}}{ds^2}} \tag{20}$$

2.2 Bending curve of the fixed beam

As an example we show a solution of the bending curve of the fixed beam, loaded by the bending moment M_o at its free end. The beam is plotted in Fig. 2 and its dimensions are given in the Table 1. If the plane, in which the bending moment M_o takes effect, contains a longitudinal axis of the beam x and the main central axis of rectangular section y or z, a plane bending occurs and the bending curve will be a planar curve. In case that the plane ν of the applied bending moment M_o will be deflected from some main central axis by the angle φ, a spatial bending will occur and the bending curve will be a spatial curve. We will solve here so far the first case of the planar bending.

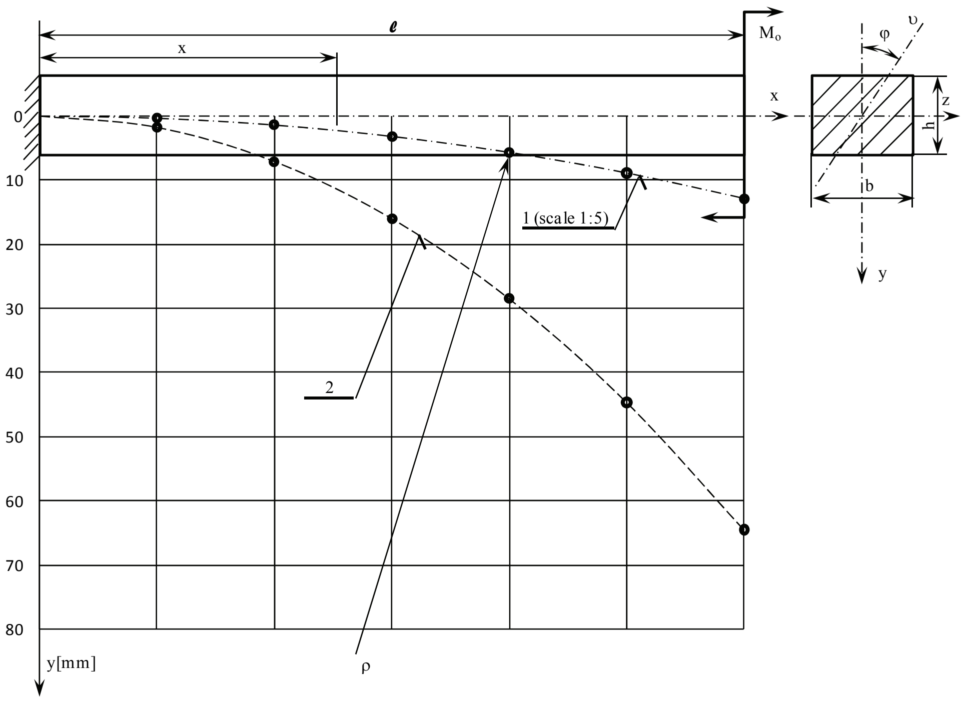

Fig. 2 Deflection curve of a fixed – end beam

The whole beam is loaded by a simple bending, the shearing force is zero. It can be assumed that Bernoulli's hypothesis concerning planarity of cross sections and also deformed beam is valid here, and then the relation [2] is valid for curvature of the beam at any of its points:

$$k = \frac{1}{\rho} = \frac{M(x)}{EI_z} \tag{21}$$

The product EI_z is called flexural rigidity of the beam.

Two equations are available for determination of curvature k at any point P of the bending curve – namely (13) and (21). Equality of the right side of these equations gives so called differential equation of the bending curve, which represents a basis for solution of deflection of the beams by analytical method:

$$\frac{y''}{\sqrt{[1+(y')^2]^3}} = \frac{M(x)}{EI_z} \tag{22}$$

Tab. 1 Parameters of loaded fixed – end beam

QUANTITY		CALCULATION	DIMENSION	SIZE
b	Width of beam cross - section		mm	16
h	Depth of beam cross – section		mm	10
l	Length of fixed – end beam		mm	600
M_o	Bending moment		Nm	100
E	Modulus of elasticity		MPa	211600
I_z	Main axial quadratic moment of cross - section	$I_z = \frac{bh^3}{12}$	mm^4	1333,3
W_o	Bending section modulus	$W_o = \frac{bh^2}{6}$	mm^3	266,6
σ_o	Maximal bending stress	$\sigma_o = \frac{M_o}{W_o}$	MPa	375
ρ	Radius of principal curvature	$\rho = \frac{EI_z}{M_o}$	mm	2821,3

In case of small flexible deformations the denominator on the left side of the equation (22) is approximately equal to one. This equation can be then written in an approximate form:

$$y'' \doteq \frac{M(x)}{EI_z} \tag{23}$$

The equation (23) is so called approximate differential equation of the bending curve and it is used in practice instead of the equation (22).

In the solved problem according to Fig. 2 the bending moment *M(x)* is constant at any point of the beam, which means in accordance with the equation (21), that curvature of the beam is constant at any of its points. The bending curve of such beam is therefore a circle.

Solution of the bending curve according to the approximate differential equation (23) is the following:

$$\left.\begin{array}{ll} EI_z y'' = M_o & 1 \cdot x = 0 \;;\; y' = 0 \;\Rightarrow\; C_1 = 0 \\ EI_z y' = M_o \cdot x + C_1 & 2 \cdot x = 0 \;;\; y = 0 \;\Rightarrow\; C_2 = 0 \\ EI_z y = M_o \frac{x^2}{2} + C_1 x + C_2 & y = \frac{M_o}{2EI_z} x^2 \end{array}\right\} \tag{24}$$

We can see that in this case the bending curve is represented by the parabola of the second degree. Assuming validity of the Bernoulli's hypothesis it is therefore possible to consider a circle to be the real bending curve. The parabola according to the solution (24) is an approximate flexural curve. One of the objectives of this paper is comparison of deflection of the beam on the basis of both solutions. Table 2 summarises the results of this comparison. The value y_p in this table means deflection expressed by the parabolic bending curve, and y_k deflection expressed by a circle. We can see that the maximum difference in deflection of the free end of the beam is 1.14%.

Tab. 2 Deflections under load of fixed – end beam

x [mm]	0	100	200	300	400	500	600
y_P [mm]	0	1,772211	7,0888	15,9499	28,3553	44,3052	63,7996
y_K [mm]	0	1,772768	7,0980	15,9985	28,4993	44,6584	64,5378
Δy [mm]	0	0,000557	0,0092	0,0486	0,1439	0,3532	0,7382
Δy [%]	0	0,03	0,13	0,25	0,50	0,79	1,14

The basis for solution was a beam that was installed in the laboratory of the Department of Mechanics of Materials at the Faculty of Mechanical Engineering of the Technical University of Mining and Metallurgy in Ostrava [3]. Its load was chosen to be close to the load of yield strength in order to obtain the maximum deflection. The bending curve in Fig. 2 is plotted as a circle (curve 1) on a scale 1:5, and also with real values of deflection curve obtained by the solution (24), (curve 2).

3 CONCLUSION

Solution of deflection of beams requires certain knowledge of mathematics, as well as the theory of elasticity. Study of the necessary chapters of mathematics requires visual geometric and physical notion of the used quantities, variables, functions, etc. The theory of elasticity uses advantageously mathematical chapters on vector and tensor calculus, theory of fields, etc. Application of the tensor calculus in theory of elasticity and plasticity is described for example in the works of the author [4], [5], [6], [7]. This paper is oriented on application of vectorial analysis in theory of elasticity, specifically on use of vector function of the scalar variable for solution of the first and second curvatures and shape of the bending curve of beams. One of the application results consists in deriving of a differential equation of the bending curve of the beams. This equation is the basis of an analytical method for solution of deflection of beams. Its simplified form of an approximate differential equation of the bending curve is used for small elastic deformations. The second objective of this paper was to demonstrate how much this approximate solution of the beam bending differs from reality. Table 2 summarises the results of solution for the specific selected fixed beam. It is obvious that in the field of small elastic deformations of beams the use of an approximate differential equation of the bending curve is in practice acceptable.

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