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CONTRIBUTION TO UNRELIABLE M/M/1/M QUEUEING SYSTEMS MODELLING

PŘÍSPĚVEK K MODELOVÁNÍ NESPOLEHLIVÝCH M/M/1/m SYSTÉMŮ HROMADNÉ OBSLUHY

Abstract

In the queueing theory we usually assume that a server breakdown can not occur (see for example in [1], [2], [3] or [4]). In other words a utility server works without failures. But in practice this assumption is not correct, the server is often a technical device and every technical device can be broken. During the breakdown the server can not work which means that the consideration of server failures has got an effect on performance measures of studied queueing system. We can say that queueing models with unreliable servers are more closely to a reality than models with reliable servers. On the other hand unreliable queueing models are more complicated. This paper presents two models of Markov queueing systems with an unreliable server.

Abstrakt

V teorii hromadné obsluhy obvykle předpokládáme, že nemůže nastat porucha obslužné linky. Jinými slovy obslužná linka pracuje bez poruch. Ale v praxi není tento předpoklad správný, obslužná linka je často technické zařízení a každé technické zařízení se může porouchat. Během poruchy obslužná linka nemůže pracovat, což znamená, že uvažování poruch obslužné linky má vliv na provozní charakteristiky studovaného systému hromadné obsluhy. Můžeme tedy říct, že modely nespolehlivých systémů hromadné obsluhy jsou mnohem blíže realitě než modely spolehlivých systémů. Na druhou stranu jsou modely nespolehlivých systémů složitější. Tento článek předkládá dva modely Markovských systémů hromadné obsluhy s nespolehlivou linkou.

1 INTRODUCTION

Assume the queueing model with a single unreliable server and with *m* places in the system, where $m \in N$, thus waiting room capacity is equal to $m-1$. Customers come to the system according to a Poisson process with rate λ, thus mean customers interarrival times are exponentially distributed with mean value equal to $\frac{1}{\lambda}$. Costumer service time is also exponential random variable with parameter μ, mean value of time needed for a customer service is equal to $\frac{1}{\mu}$. According to extended Kendall‘s notation it is M/M/1/*m* queueing system.

Server failures occur according to a Poisson process with rate $\overline{\lambda}$, time between failures is an exponential random variable with mean value equal to $\frac{1}{\overline{\lambda}}$. Notice that we do not consider a possibility of several failures occurrence at the same time. That means that after occurrence of breakdown another failure can not occur until the server is broken. Time to repair is exponentially distributed with parameter $\overline{\mu}$, mean time to repair is equal to $\frac{1}{\overline{\mu}}$.

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Let us discuss the service discipline and customers behavior. Costumers are served one by one according to FIFO (First In - First Out) discipline. This paper presents:

- model of queueing system with customer service repeating,
- model of queueing system with mass customers outgoing after breakdown occurrence.

2 MODEL OF QUEUEING SYSTEM WITH COSTUMER SERVICE REPEATING

At the moment of breakdown occurrence may occur two different states (if there is a costumer in the service just at the moment):

- if there is less than $m-1$ customers in queue, costumer comes back to first queue place and after repair of server its service starts from the beginning,
- if there is exactly $m-1$ customers in queue, costumer leaves the system and we consider him as rejected.

Let illustrate our model graphically as a state transition diagram (see in fig. 1). Nodes represent particular system states and arrows indicate possible transitions with corresponding rate. Notice that the graph is drawn without loops.

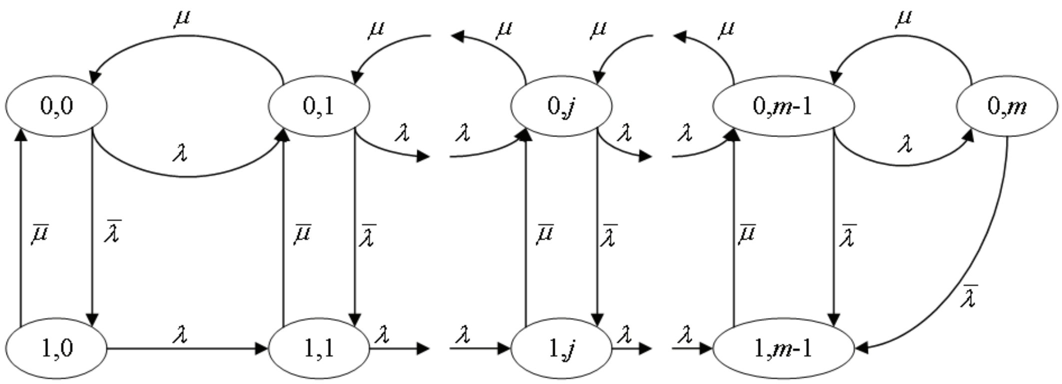

Fig. 1 State transition diagram of unreliable M/M/1/*m* queueing system with costumer service repeating

States of the model can be divided into two groups:

- states denoted as pair (0,*j*), where the first symbol 0 says that the server is failure free and the second symbol *j* represents number of costumers in the system,
- states denoted as pair (1,*j*), where the first symbol 1 means server breakdown and the second symbol *j* is number of waiting customers.

On the basis of state transition diagram we can obtain finite system of differential equations for probabilities of particular states depending on time *t* (see common method in [1]):

$$P'_{(0,0)}(t) = -(\lambda + \bar{\lambda})P_{(0,0)}(t) + \mu P_{(0,1)}(t) + \bar{\mu} P_{(1,0)}(t),$$

$$P'_{(0,j)}(t) = \lambda P_{(0,j-1)}(t) - (\lambda + \mu + \bar{\lambda})P_{(0,j)}(t) + \mu P_{(0,j+1)}(t) + \bar{\mu} P_{(1,j)}(t) \text{ for } j = 1,2,\dots,m-1,$$

$$P'_{(0,m)}(t) = \lambda P_{(0,m-1)}(t) - (\mu + \bar{\lambda})P_{(0,m)}(t),$$

$$P'_{(1,0)}(t) = \bar{\lambda} P_{(0,0)}(t) - (\lambda + \bar{\mu})P_{(1,0)}(t),$$

$$P'_{(1,j)}(t) = \bar{\lambda} P_{(0,j)}(t) + \lambda P_{(1,j-1)}(t) - (\lambda + \bar{\mu})P_{(1,j)}(t) \text{ for } j = 1,2,\dots,m-2,$$

$$P'_{(1,m-1)}(t) = \bar{\lambda} P_{(0,m-1)}(t) + \bar{\lambda} P_{(0,m)}(t) + \lambda P_{(1,m-2)}(t) - \bar{\mu} P_{(1,m-1)}(t)$$

with the condition $\sum_{j=0}^{m} P_{(0,j)}(t) + \sum_{j=0}^{m-1} P_{(1,j)}(t) = 1$.

$P_{(i,j)}(t)$, where $i \in \{0;1\}$ and $j \in \{0;1;...;m\}$, is the probability that studied queueing system is found in state (i,j) in time t.

For $t \to \infty$ we get system of linear equations for steady state probabilities that are not dependent on time t. For this queueing system we obtain $2m + 1$ linear equations:

$$0 = -(\lambda + \bar{\lambda})P_{(0,0)} + \mu P_{(0,1)} + \bar{\mu} P_{(1,0)},$$

$$0 = \lambda P_{(0,j-1)} - (\lambda + \mu + \bar{\lambda})P_{(0,j)} + \mu P_{(0,j+1)} + \bar{\mu} P_{(1,j)} \text{ for } j = 1,2,...,m-1,$$

$$0 = \lambda P_{(0,m-1)} - (\mu + \bar{\lambda})P_{(0,m)},$$

$$0 = \bar{\lambda} P_{(0,0)} - (\lambda + \bar{\mu})P_{(1,0)},$$

$$0 = \bar{\lambda} P_{(0,j)} + \lambda P_{(1,j-1)} - (\lambda + \bar{\mu})P_{(1,j)} \text{ for } j = 1,2,...,m-2,$$

$$0 = \bar{\lambda} P_{(0,m-1)} + \bar{\lambda} P_{(0,m)} + \lambda P_{(1,m-2)} - \bar{\mu} P_{(1,m-1)}$$

with normalization equation $\sum_{j=0}^{m} P_{(0,j)} + \sum_{j=0}^{m-1} P_{(1,j)} = 1$.

By numerical solving of the equations system in Matlab we obtain stationary probabilities. On the basis of stationary probabilities we are able to compute performance measures of studied system. Let us focused on two measures - mean number of customers in the service denoted as *ES* and mean number of waiting customers *EL*. These measures we can compute according to formula for mean value computation of discrete random variable. We get:

$$ES = \sum_{j=1}^{m} P_{(0,j)} \text{ and } EL = \sum_{j=2}^{m} (j-1)P_{(0,j)} + \sum_{j=1}^{m-1} jP_{(1,j)}.$$

3 MODEL OF QUEUEING SYSTEM WITH MASS CUSTOMERS DEPARTURE AFTER BREAKDOWN OCCURRENCE

In this case we consider that breakdown occurrence causes departure of all costumers from the system (we consider these costumers are rejected). Further we assume that incoming customers are rejected until server is not repaired. Relevant state transition diagram you can see in fig. 2.

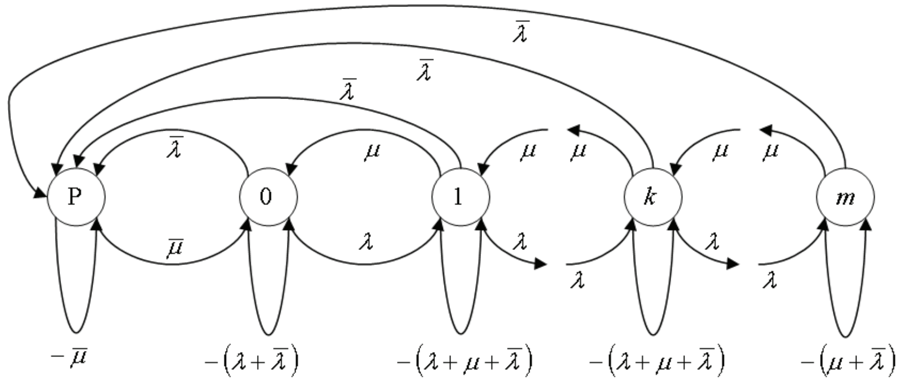

Fig. 2 State transition diagram of unreliable M/M/1/*m* queueing system with mass customers departure after breakdown occurrence

In this case we need not to use two-valued code for system states description. To states corresponding to reliable queueing system (states denoted as 0,1,…,*m*) the state denoted as P is attached. State P means server breakdown.

Finite system of differential equations is shown below:

$$P_P'(t) = -\overline{\mu} P_p(t) + \overline{\lambda} \sum_{k=0}^{m} P_k(t),$$

$$P_0'(t) = \overline{\mu} P_p(t) - \left(\lambda + \overline{\lambda}\right) P_0(t) + \mu P_1(t),$$

$$P_k'(t) = \lambda P_{k-1}(t) - \left(\lambda + \mu + \overline{\lambda}\right) P_k(t) + \mu P_{k+1}(t) \text{ for } k = 1,2,...,m-1,$$

$$P_m'(t) = \lambda P_{m-1}(t) - \left(\mu + \overline{\lambda}\right) P_m(t),$$

with the condition $P_p(t) + \sum_{k=0}^{m} P_k(t) = 1$.

For $t \to \infty$ we obtain *m* + 1 linear equations:

$$0 = -\overline{\mu} P_p + \overline{\lambda} \sum_{k=0}^{m} P_k ,$$

$$0 = \overline{\mu} P_p - \left(\lambda + \overline{\lambda}\right) P_0 + \mu P_1 ,$$

$$0 = \lambda P_{k-1} - \left(\lambda + \mu + \overline{\lambda}\right) P_k + \mu P_{k+1} \text{ for } k = 1,2,...,m-1 ,$$

$$0 = \lambda P_{m-1} - \left(\mu + \overline{\lambda}\right) P_m$$

with normalization equation $P_p + \sum_{k=0}^{m} P_k = 1$. From first and normalization equation we can derive formula for P_p computation:

$$P_p = \frac{\overline{\lambda}}{\overline{\lambda} + \overline{\mu}} .$$

Performance measures under consideration we can compute:

$$ES = \sum_{k=1}^{m} P_k \text{ and } EL = \sum_{k=2}^{m} (k-1) P_k .$$

4 EXECUTED EXPERIMENTS AND THEIR OUTCOMES

Let consider unreliable M/M/1/5 system (maximal queue length is 4 costumers). Let consider 3 constant system parameters - $\lambda = 9\,h^{-1}$, $\mu = 10\,h^{-1}$ and $\overline{\mu} = 0{,}2\,h^{-1}$. The last parameter $\overline{\lambda}$ will be increased from the minimum value $\overline{\lambda} = 0{,}0001\,h^{-1}$ (this value corresponds to mean time between failures equal to $10\,000\,h$) to maximum value $\overline{\lambda} = 0{,}1\,h^{-1}$ (mean time between failures is $10\ h$).

Steady state probabilities for first model we obtain by solution of the equation system given below in fig. 3.

$$\left|\begin{array}{ccccccccccc} -(9+\bar{\lambda}) & 10 & 0 & 0 & 0 & 0 & 0{,}2 & 0 & 0 & 0 & 0 \\ 9 & -(19+\bar{\lambda}) & 10 & 0 & 0 & 0 & 0 & 0{,}2 & 0 & 0 & 0 \\ 0 & 9 & -(19+\bar{\lambda}) & 10 & 0 & 0 & 0 & 0 & 0{,}2 & 0 & 0 \\ 0 & 0 & 9 & -(19+\bar{\lambda}) & 10 & 0 & 0 & 0 & 0 & 0{,}2 & 0 \\ 0 & 0 & 0 & 9 & -(19+\bar{\lambda}) & 10 & 0 & 0 & 0 & 0 & 0{,}2 \\ 0 & 0 & 0 & 0 & 9 & -(10+\bar{\lambda}) & 0 & 0 & 0 & 0 & 0 \\ \bar{\lambda} & 0 & 0 & 0 & 0 & 0 & -9{,}2 & 0 & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 & 0 & 0 & 9 & -9{,}2 & 0 & 0 & 0 \\ 0 & 0 & \bar{\lambda} & 0 & 0 & 0 & 0 & 9 & -9{,}2 & 0 & 0 \\ 0 & 0 & 0 & \bar{\lambda} & 0 & 0 & 0 & 0 & 9 & -9{,}2 & 0 \\ 0 & 0 & 0 & 0 & \bar{\lambda} & \bar{\lambda} & 0 & 0 & 0 & 9 & -0{,}2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right| \cdot \left|\begin{array}{c} P_{0,0} \\ P_{0,1} \\ P_{0,2} \\ P_{0,3} \\ P_{0,4} \\ P_{0,5} \\ P_{1,0} \\ P_{1,1} \\ P_{1,2} \\ P_{1,3} \\ P_{1,4} \end{array}\right| = \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}\right|$$

Fig. 3 Equation system for first model (matrix notation)

Equation system for second model solution is shown in fig. 4.

$$\left|\begin{array}{ccccccc} -0{,}2 & \bar{\lambda} & \bar{\lambda} & \bar{\lambda} & \bar{\lambda} & \bar{\lambda} & \bar{\lambda} \\ 0{,}2 & -(9+\bar{\lambda}) & 10 & 0 & 0 & 0 & 0 \\ 0 & 9 & -(19+\bar{\lambda}) & 10 & 0 & 0 & 0 \\ 0 & 0 & 9 & -(19+\bar{\lambda}) & 10 & 0 & 0 \\ 0 & 0 & 0 & 9 & -(19+\bar{\lambda}) & 10 & 0 \\ 0 & 0 & 0 & 0 & 9 & -(19+\bar{\lambda}) & 10 \\ 0 & 0 & 0 & 0 & 0 & 9 & -(10+\bar{\lambda}) \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right| \cdot \left|\begin{array}{c} P_P \\ P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array}\right| = \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}\right|$$

Fig. 4 Equation system for second model (matrix notation)

Establish $\bar{\rho} = \dfrac{\bar{\lambda}}{\bar{\mu}}$ as breakdowns load. Focus on dependence of selected performance measures on $\bar{\rho}$. These graphical relations we can see in fig. 5 and fig. 6. Outcomes obtained by analytic computation are supplemented by outcomes gained by simulation of studied system in software Witness. All simulation experiments were executed for 2 years of real time. The constant curve shown in both graphs reflects reliable system performance measure behavior.

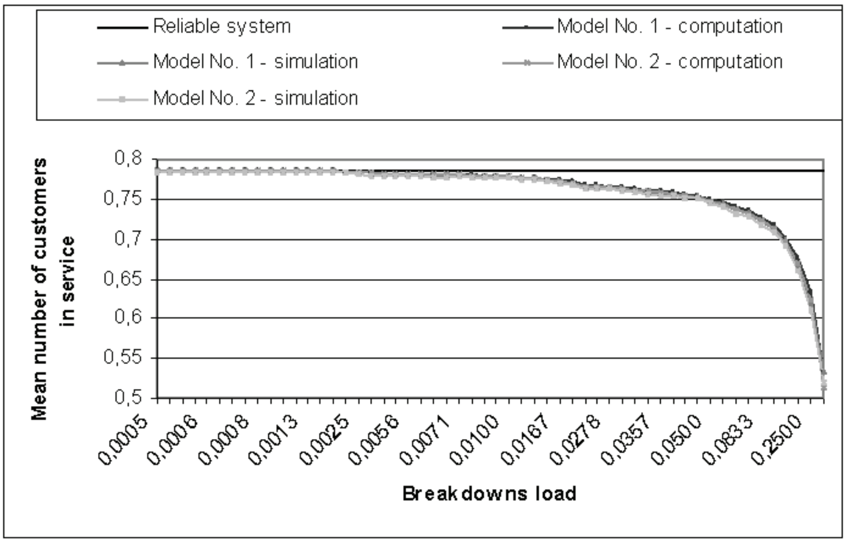

Fig. 5 Graphical dependence of ES on $\bar{\rho}$.

On the basis of fig. 5 we can say that behavior of all models is practically the same. Approximately for $\overline{\rho} \leq 0{,}01$ there is an insignificant difference between reliable and unreliable system. We can say that for $\overline{\rho} \leq 0{,}01$ model of unreliable system could be replaced by model of reliable system without cardinal outcomes differences. On the other hand for $\overline{\rho} > 0{,}01$ curve of mean number of costumers in service for unreliable system decreases against reliable system.

Fig. 6 reveals different behavior of first model and second model for higher $\overline{\rho}$. Once again approximately for $\overline{\rho} \leq 0{,}01$ there is not remarkable divergence between both models and between reliable and unreliable system. But for $\overline{\rho} > 0{,}01$ mean number of waiting customers for first model increases and for second model decreases. Notice that there are no significant differences between analytic and simulation outcomes of appropriate models.

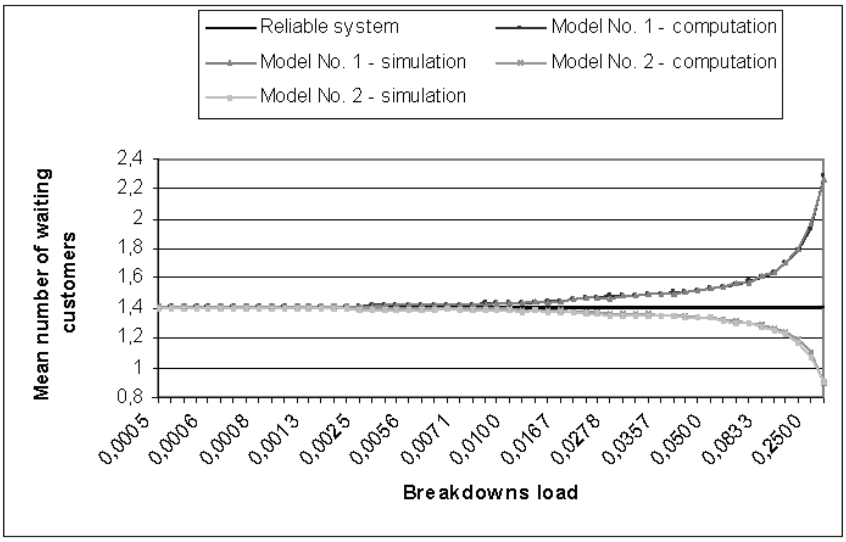

Fig. 6 Graphical dependence of EL on $\overline{\rho}$

5 CONCLUSIONS

This paper presents models of two unreliable single server queueing systems. These models vary in behavior of customers after breakdown occurrence. The major part of paper is focused on mathematical models of studied systems. In the end of paper there are shown relations of two selected performance measures on defined parameter $\overline{\rho}$.

On the basis of executed experiments we can say that there are no significant differences in outcomes between mathematical and simulation model of studied queueing systems. Presented Markov queueing models can serve as a guidance for creation of more difficult models for example with several servers.

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