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Pavel MACURA[*], František FOJTÍK[**]

TENSOR VALUES OF THE SECOND ORDER IN THEORY OF ELASTICITY

TENZOROVÉ VELIČINY DRUHÉHO ŘÁDU V NAUCE O PRUŽNOSTI A PEVNOSTI

Abstract

The paper is devoted to basic concepts and quantities that are used in theory of elasticity. All these quantities are tensor quantities of various orders – scalars as tensors of zero order, vectors as tensors of the first order and tensors of the second order, named usually tensors. Tensors of higher orders are defined and also used in theory of elasticity. Main interest of the paper is focused on tensors of the second order, since this category comprises some basic quantities used in theory of elasticity. These are mainly a stress tensor, strain tensor and tensor of quadratic moments of area. Physical meaning and geometric notion of scalars and vectors is comparatively easy, in case of tensors of the second order it is somewhat more difficult and it is usually not mentioned in textbooks and publications. The authors attempt in their paper to demonstrate a clear geometric notion and physical meaning of tensor quantities of the second order on examples of stress tensor, strain tensor and namely tensor of quadratic moment of area.

Abstrakt

Příspěvek je věnován základním pojmům a veličinám, se kterými se operuje a počítá v nauce o pružnosti a pevnosti. Všechny tyto veličiny jsou tenzorové veličiny různých řádů – skaláry coby tenzory nultého řádu, vektory jako tenzory prvního řádů a tenzory druhého řádů, označované zpravidla jako tenzory. Jsou definovány a v nauce o pružnosti a pevnosti se počítá i s tenzory vyšších řádů. V příspěvku je hlavní pozornost věnována tenzorům druhého řádu, protože do této kategorie tenzorů patří některé základní veličiny, se kterými se v nauce o pružnosti a pevnosti počítá. Jsou to hlavně tenzor napjatosti, tenzor deformace a tenzor kvadratických momentů plochy. Fyzikální význam a geometrická představa skalárů a vektorů je poměrně snadná, u tenzorů druhého řádu je poněkud obtížnější a nebývá v učebnicích a publikacích uváděna. Cílem předloženého příspěvku je pokusit se ukázat názornou geometrickou představu a fyzikální význam tenzorových veličin druhého řádu na příkladech tenzoru napjatosti, tenzoru deformace a hlavně tenzoru kvadratického momentu plochy.

1 INTRODUCTION

Three groups of equations are available for solution of tasks of theory of elasticity – stress equations of equilibrium, geometric equations and physical equations – determining dependencies among the three physical quantities, i.e. among the displacement vector, stress tensor and strain tensor. Displacement vector as tensor of the first order is given by three scalar coordinates, stress tensors and strain tensors as the tensors of the second order are determined by nine scalar coordinates or possibly by three vector coordinates. Due to the fact that stress tensors and strain tensors are symmetrical, only six scalar coordinates suffice for their unequivocal determination. The state in a certain point of a loaded body is therefore given by fifteen scalar coordinates of physical quantities of displacement vector and stress and strain tensors. Fifteen equations of theory of elasticity are available for their determination.

[*] prof. Ing. DrSc., Department of Mechanics of Materials, Faculty of Mechanical Engineering, VŠB – TU Ostrava, 17. listopadu 15, tel. (+420) 597323598, e-mail: pavel.macura@vsb.cz

[**] Ing. Ph.D., Department of Mechanics of Materials, Faculty of Mechanical Engineering, VŠB – TU Ostrava, 17. listopadu 15, tel. (+420) 597323292, e-mail: frantisek.fojtik@vsb.cz

At solution of components subjected to bending stress and torque another quantity appears and is used, namely quadratic moment of area of cross section of this component. This quantity belongs to the group of tensor quantities of the second order. Together with the stress tensor and strain tensor these are three the most frequently used tensors of the second order in theory of elasticity. Special tensor calculus was developed in mathematics for calculations with tensors and it is possible to calculate by it with tensors without knowing their physical significance. For understanding of tensor calculus it is, however, appropriate to make a clear geometric concept of some tensor of the second order. The three above mentioned tensors, analysed in this paper, can serve this purpose.

2 TENSOR QUANTITIES OF THE SECOND ORDER

One of possible definitions of a tensor of the second order is with use of dyads [1]. Dyad is a quantity, which has apart from the absolute value generally two directions in the given order. The tensor calculus defines apart from scalar and vector product also a dyadic product of two vectors, which is a quantity, the absolute value of which is equal to a product of absolute values of both vectors and the directions of which are directions of both vectors in the given order. The dyad is therefore formed by dyadic product of two vectors and the tensor is then formed and defined as a set of *n* dyads. It is then possible to form a clear geometric concept of the tensor of the second order on the basis of fact, which vectors form individual dyads in the set of dyads of investigated tensor of the second order.

Tensor of the second order can form any number of dyads, it is possible in a 3D space to transform them to a set of three dyads only, such tensor is called complete. The tensor can be in some cases given by the set of only two dyads, in that case this is a planar tensor. The simplest tensor of the second order is formed just by one dyad, it is linear tensor. In theory of elasticity to these simplest tensors correspond for example planar or linear state of stress.

2.1 Stress tensor

Perhaps the most illustrative geometric concept of the tensor of the second order can be made on the example of stress tensor. Theory of elasticity uses for stress analysis in a certain point of loaded body the method of section. Hypothetical planar section is lead through the chosen point of the body, which divides the body into two parts. Figure 1 shows the section lead through the point T of the fixed beam.

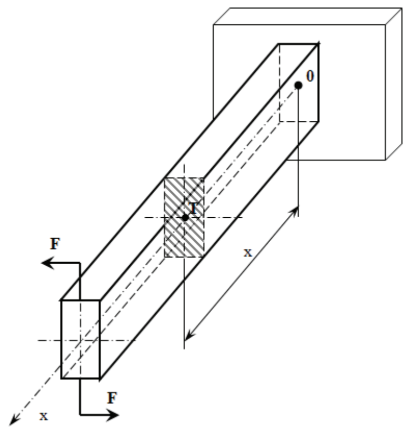

Fig. 1 Method of sections for analysis at loaded body

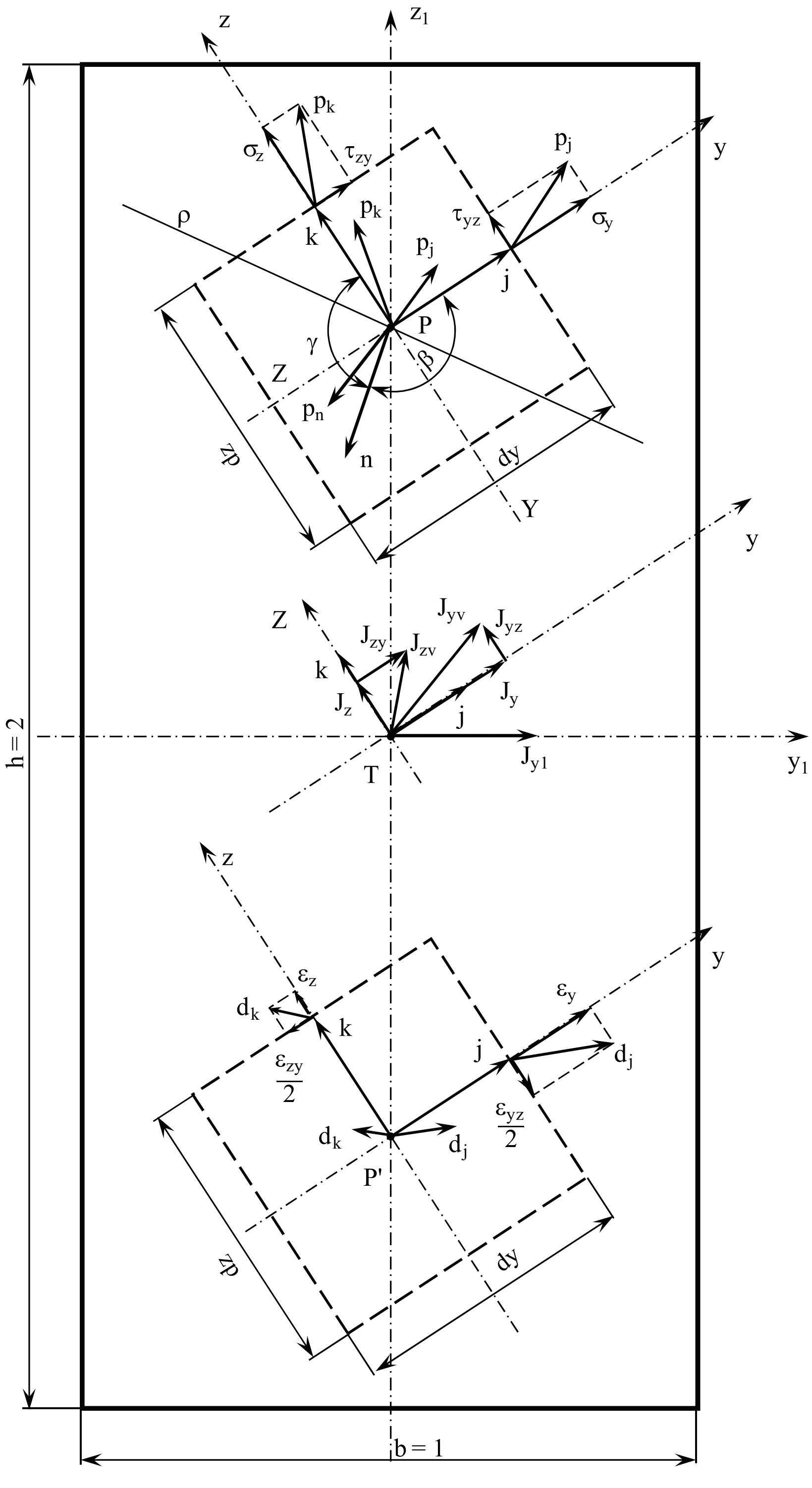

Fig. 2 The stress, strain and quadratic moment of area tensors

On this plane of the section an arbitrary point P is chosen (Fig. 2), in neighbourhood of which the state of stress will be analysed. The point P can be envisioned in a form of infinitely small cube with dimensions dx, dy and dz. Let us first assume that planar state of stress will be created at the point P and that the main stress $\boldsymbol{\sigma_1}$ in direction of the axis x, perpendicular to the wall of the small cube dy×dz, will be zero. Normal stress $\boldsymbol{\sigma_y}$ and shear stress $\boldsymbol{\tau_{yz}}$, will affect the wall of the small cube dx×dz, on the wall dx×dy is the normal stress $\boldsymbol{\sigma_z}$ and shear stress $\boldsymbol{\tau_{zy}}$. Stress on individual walls of the small cube can be vectorially added up and we will get the resulting stresses on these walls $\mathbf{p_j}$ and $\mathbf{p_k}$.

Stress tensor at the point P is then given by the sum of two dyads, which form dyadic products of vectors $\mathbf{p_j}$ and unit normal vector **j** to the wall of the small cube dx×dz (to the plane Y) and vectors $\mathbf{p_k}$ and unit normal vector **k** to the wall of the small cube dx×dy (to the plane Z).

$$\begin{aligned} \mathbf{T}_{\boldsymbol{\sigma}} = \mathbf{p}_j\mathbf{j} + \mathbf{p}_k\mathbf{k} &= \left(\sigma_y\mathbf{j} + \tau_{yz}\mathbf{k}\right)\mathbf{j} + \left(\tau_{zy}\mathbf{j} + \sigma_z\mathbf{k}\right)\mathbf{k} = \\ &= \sigma_y\mathbf{jj} + \tau_{zy}\mathbf{jk} + \\ &+ \tau_{yz}\mathbf{kj} + \sigma_z\mathbf{kk} \end{aligned} \tag{1}$$

It can be seen from the equation (1), that it is possible to write down the same tensor $\mathbf{T}_{\boldsymbol{\sigma}}$ by two dyads or four dyads. The vectors $\mathbf{p_j}$ and $\mathbf{p_k}$ are called vectorial coordinates of the tensor, scalars σ_y, τ_{zy}, τ_{yz} and σ_z are called numeric coordinates of the stress tensor. Notation of the tensor according to the equation (1) is comparatively laborious, that's why it is usually entered in the form:

$$\mathbf{T}_{\boldsymbol{\sigma}} = \begin{Vmatrix} \sigma_y & \tau_{zy} \\ \tau_{yz} & \sigma_z \end{Vmatrix} \tag{2}$$

Notation according to the equation (2) means that there exists one unequivocal representation of the set of all tensors of the second order to the set of square matrices. Matrix in the equation (2) is called matrix image of the tensor $\mathbf{T}_{\boldsymbol{\sigma}}$ or shortly matrix of the tensor $\mathbf{T}_{\boldsymbol{\sigma}}$.

It is then possible to calculate with tensors and vectors in the tensor calculus similarly as with matrices in the matrix calculus [2]. Some mathematical operations between tensors or tensors and vectors have also clear geometric and physical meaning. For example result of scalar product of the stress tensor $\mathbf{T}_{\boldsymbol{\sigma}}$ and unit normal vector **n** of the plane ρ, turned in respect to the coordinate system y and z by the angles β and γ, is the vector of final stress $\mathbf{p_n}$ on the plane ρ :

$$\mathbf{T}_{\boldsymbol{\sigma}}.\,\mathbf{n} = \begin{Vmatrix} \sigma_y & \tau_{zy} \\ \tau_{yz} & \sigma_z \end{Vmatrix} . \begin{Vmatrix} \cos\beta \\ \cos\gamma \end{Vmatrix} = \begin{Vmatrix} \sigma_y \cos\beta + \tau_{zy}\cos\gamma \\ \tau_{yz}\cos\beta + \sigma_z\cos\gamma \end{Vmatrix} = \mathbf{p_n} \tag{3}$$

Dyadic product of vectors $\mathbf{p_n}$ and **n** then form a dyad on the turned plane ρ, passing through the investigated point P of the loaded body. It is possible to make the analysis of state of stress at the point P also with use of Mohr circle of state of stress, as it is shown in the Fig. 3.

Use of tensor calculus at the analysis of state of stress in theory of elasticity significantly accelerates calculations and makes notation of the used equations more transparent. The work [3] deals in greater detail with the complete stress tensor for the case of 3D state of stress. Moreover it analyses main directions of the tensor of the second order and calculates main stresses, determines invariants of the stress tensor, explains notions of spherical tensor and deviator, determines invariants of stress deviator and defines with use of them important quantities, such as stress intensity or shear stress intensity, which have great significance in theory of elasticity and plasticity.

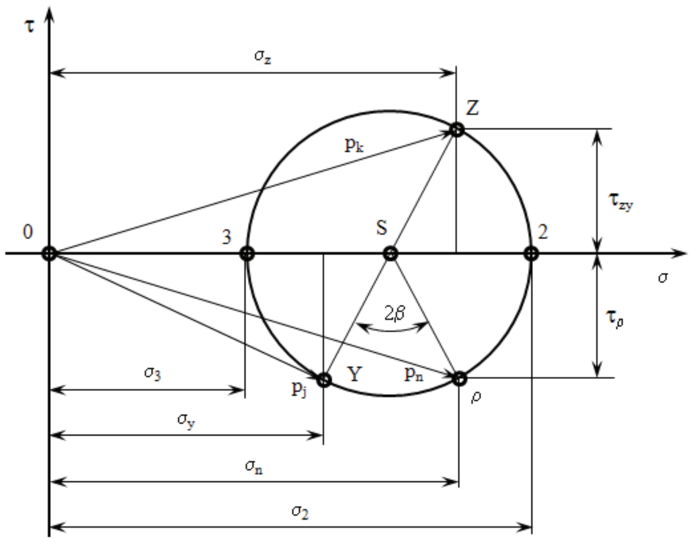

Fig. 3 Stress analysis by means of Mohr's circle of stresses

2.2 Strain tensor

Elastic body deforms as a result of load. At the same time individual points of the body are displaced. This change of position of the points of the body can be characterised by displacement vectors **u**, **v** and **w**. The Figure 2 shows also an analysis of the state of strain in the neighbourhood of the displaced point P′. A planar deformation is assumed at this point (but also 3D state of stress), when relative elongation in direction of the coordinate axis x is zero. In the environment of the point P′ the original infinitely small cube is plotted with dimensions dx, dy and dz and two unit vectors **j** and **k** in direction of axes of the initial coordinate system y and z. This small cube deforms as a result of load, its walls recede or approach and gets mutually bevelled. Magnitude of deformation can be then characterised by vectors of relative elongation **ε** and shearing strain **γ**. Original unit vector **j** will be elongated by the relative elongation ε_y, unit vector **k** will change its length by ε_z, and original right angle between the vectors **j** and **k** will change by the angle γ_{yz}. End points of the vector **j** and **k** will then take new positions, given by vector sums of deformations **ε** and **γ**. Vectors of final deformation $\mathbf{d}_\mathbf{j}$ and $\mathbf{d}_\mathbf{k}$ will result from these vector sums.

Dyadic products of the vectors **j** and $\mathbf{d}_\mathbf{j}$ and **k** and $\mathbf{d}_\mathbf{k}$ form two dyads, the set of which is defined by the strain tensor:

$$\begin{aligned}\mathbf{T}_\varepsilon &= \mathbf{d}_\mathbf{j}\mathbf{j} + \mathbf{d}_\mathbf{k}\mathbf{k} = \left(\varepsilon_y \mathbf{j} + \frac{\gamma_{yz}}{2}\mathbf{k}\right)\mathbf{j} + \left(\frac{\gamma_{zy}}{2}\mathbf{j} + \varepsilon_z \mathbf{k}\right)\mathbf{k} = \\ &= \varepsilon_y \mathbf{jj} + \frac{\gamma_{zy}}{2}\mathbf{jk} + \\ &+ \frac{\gamma_{yz}}{2}\mathbf{kj} + \varepsilon_z \mathbf{kk}\end{aligned} \tag{4}$$

Similarly as in case of the stress tensor the vectors $\mathbf{d}_\mathbf{j}$ and $\mathbf{d}_\mathbf{k}$ are called vector coordinates and scalars ε_y, $\frac{\gamma_{zy}}{2}$, $\frac{\gamma_{yz}}{2}$ and ε_z numeric coordinates of the strain tensor. Notation of this tensor is then also simplified in the form of matrix:

$$\mathbf{T}_\varepsilon = \begin{Vmatrix} \varepsilon_y & \dfrac{\gamma_{zy}}{2} \\ \dfrac{\gamma_{yz}}{2} & \varepsilon_z \end{Vmatrix} \tag{5}$$

Similarly as in case of all tensors of the second order it is possible to determine for the strain tensor as well its main directions, magnitude of main deformations, invariants of strain tensor and deviator and values of the strain intensity, or intensity of shearing strains. The work [4] describes the strain tensor similarly. It analyses relations between components of the strain tensor and displacement vector, as well as dependencies at turning of the coordinate system both analytically and graphically with use of the Mohr circle for deformations.

2.3 Quadratic moments of area

Another quantity of theory of elasticity, which belongs to the group of tensors of the second order, are quadratic moments of section area of the loaded body. Quadratic moments are related not only to areas, but also to lines, or volumes of the bodies. Quadratic moments of area are sometime incorrectly called inertia moments of area due to their similarity with definition of the inertia moment of material body.

The work [5] has analysed in detail quadratic moments of abscissa, this paper will concentrate on quadratic moments of area, used at calculations of stress at load of the bodies by bending and torque.

Quadratic moments of area are defined in the following manner (Fig. 4) :

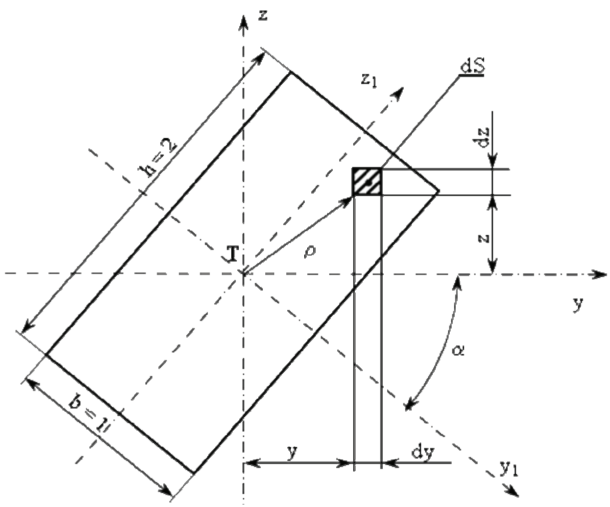

Fig. 4 Calculation of quadratic moments of area

a) $J_y = \int_{(S)} z^2 dS$ . . . . axial quadratic moment to the axis y (6)

b) $J_z = \int_{(S)} y^2 dS$ . . . . axial quadratic moment to the axis z (7)

c) $J_{yz} = \int_{(S)} yz dS$ . . . .deviation moment to the axes y and z (8)

d) $J_p = \int_{(S)} \rho^2 dS$ . . . . polar quadratic moment to the point T (9)

Calculation of quadratic moments of area to the new system of coordinates, turned in respect to the initial system y, z by the angle α, gives similar expressions as for the quantity of stress and strain [6] and as for quadratic moments of abscissa [5]. These are therefore the tensor quantities of the second order.

Axial quadratic moment of area can be represented by the vector, collinear to the axis, to which it is calculated. Deviation moment of area is then represented by the vector, collinear to the second coordinate axis. Vector sum of these two vectors gives the „resulting“ quadratic moment of area to the chosen axis. The Fig. 2 shows at the point T the vectors, corresponding to axial, deviation and resulting quadratic moments of section area in respect to the axes of the coordinate system y and z. These axes are collinear with individual unit vectors **j** and **k**.

Dyadic products of the vectors $\mathbf{J_{yv}}$ and **j** and $\mathbf{J_{zv}}$ and **k** form dyads, the set of which gives the tensor of quadratic moment of section area of the body. Unit vectors **j** and **k** in these dyads characterise position of axes, to which the resulting quadratic moments of area $\mathbf{J_{yv}}$ and $\mathbf{J_{zv}}$ are related.

It is therefore possible to write this tensor in the following form:

$$\begin{aligned} \mathbf{T} &= \mathbf{J_j j} + \mathbf{J_k k} = \left(J_y \mathbf{j} + J_{yz}\mathbf{k}\right)\mathbf{j} + \left(J_{zy}\mathbf{j} + J_z \mathbf{k}\right)\mathbf{k} = \\ &= J_y \mathbf{jj} + J_{zy}\mathbf{jk} + \\ &+ J_{yz}\mathbf{kj} + J_z \mathbf{kk} \end{aligned} \tag{10}$$

Similarly as in case of the stress tensor it is here also possible to use the shortened notation of the tensor in the equation (10) in the form of matrix:

$$\mathbf{T} = \begin{Vmatrix} J_y & J_{zy} \\ J_{yz} & J_z \end{Vmatrix} \tag{11}$$

With use of the tensor calculus it is then possible to apply the necessary mathematical operations between tensors and vectors. This is used in practice for example at solution of the size of quadratic moments of area in respect to the turned axes and particularly to their extreme values, called main quadratic moments of section area, which are related to the main axes. Calculation of extreme values and positions of main axes can be made subject to the condition that result of scalar product of the tensor of quadratic moment of area **T** and unit vector **n** in direction of the main axis must be the vector, collinear with the vector **n** :

$$\mathbf{T}\,.\,\mathbf{n} = \lambda\,\mathbf{n} \tag{12}$$

Solution of the equation (12) for quadratic moments of area is similar as for quadratic moments of abscissa, which was described in detail in the work [5]. It is possible to derive at it invariants of the tensor of quadratic moment of area and express this tensor by the main quadratic moments of area:

$$\mathbf{T} = \begin{Vmatrix} J_{y1} & 0 \\ 0 & J_{z1} \end{Vmatrix} \tag{13}$$

Main quadratic moments of section area have great significance at projects of beams, when the efforts are focused on ensuring the biggest quadratic moment of section area and thus the maximum load capacity with the lightest possible mass of the beam. This is the reason for rolling of the sections I, H and other.

Quadratic moments of area in respect to the turned axes and main quadratic moments of area can also be resolved graphically with use of the circle for quadratic moments of area, as it is shown in the Fig. 5.

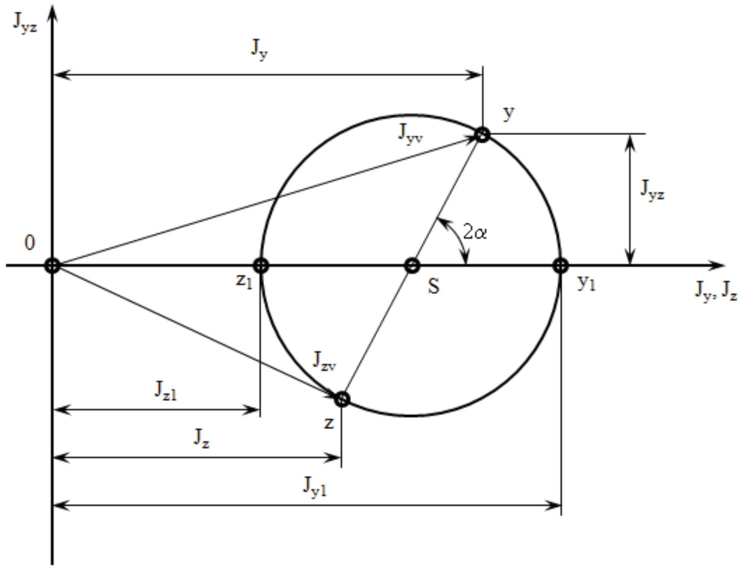

Fig. 5 Circle for the solution of quadratic moments of area

3 CONCLUSIONS

At calculations in theory of elasticity and also in theory of plasticity it is possible to apply with an advantage the tensor calculus. For its understanding it is appropriate to make a clear geometric concept and physical significance of some tensor quantity of the second order. Authors describe in this paper his geometric idea of three basic quantities, used in theory of elasticity. These quantities are mechanical stress and strain at the point of the loaded body and quadratic moments of section area of the body. This is derived from definition of the tensor of the second order with use of dyads. Dyads form arranged pairs of vectors and namely these pairs of vectors have their physical meaning at the analysed tensors of the second order. In case of the stress tensor these vectors are represented by the vector of resulting stress on certain plane, characterised by the unit normal vector to this plane, which is the second vector of this dyad. Such dyad therefore defines unequivocally a position of the plane and magnitude and direction of the resulting stress to it at certain point of the loaded body. However, for unequivocal determination of state of stress at this point it is necessary to know the magnitudes and directions of the final stress in three planes, the general stress tensor is therefore determined by the set of three dyads. Vectors of resulting stress in three planes form vector coordinates of the stress tensor, their components in direction of axes of the coordinate system then form scalar coordinates of the stress tensor.

Similarly as in case of the strain tensor the dyads are formed by unit vectors of coordinate system and by resulting strain vectors of these unit vectors.

In case of the tensor of quadratic moment of area the dyads form unit vectors in direction of axes, to which the quadratic moments of area are calculated, and magnitudes of resulting quadratic moments of section area to these axes.

Calculation of magnitude of all three quantities in respect to the turned coordinate system results in similar equations, which differ only by notation of the investigated quantity. This makes it possible to abstract from this quantity and to create a general tensor calculus for calculation with tensors of the second and higher orders.

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