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DESIGN OF DISCRETE-TIME COMPENSATOR FOR REFERENCE TRACKING WITH DISTURBANCE REJECTION

NÁVRH DISKRÉTNEHO KOMPENZÁTORA NA SLEDOVANIE REFERENČNÉHO SIGNÁLU A POTLAČENIE PORUCHY

Abstract

The paper deals with the design of a discrete-time state space compensator to track a reference command when the plant is subject to a measurable disturbance signal. The approach called Command Generator Tracker [3] covers a broad class of reference and disturbance signals describable by linear differential equations with constant coefficients. The design technique is LQ based and its principle consists in augmenting the plant model so that the resulting compensator includes modes of both the reference and disturbance signals. In such a case, proper reference tracking and disturbance rejection are guaranteed. Standard LQ problem solution applied for the extended system yields both the feedback and the feedforward gains of the CGT compensator.

Abstrakt

Príspevok sa zaoberá návrhom diskrétneho stavového kompenzátora, ktorý zabezpečí sledovanie referenčného signálu v prípade, keď na riadený systém pôsobí merateľná porucha. Kompenzátor navrhnutý použitím CGT prístupu [3] dokáže zabezpečiť splnenie požiadaviek návrhu pre širokú triedu signálov, ktoré je možné opísa lineárnymi diferenciálnymi rovnicami s konštantnými koeficientami. Kompenzátor sa navrhuje ako štandardný LQ regulátor pre rozšírený systém tak, aby zahrňoval módy referenčného a poruchového signálu. V takom prípade je zaručené asymptotické sledovanie a potlačenie poruchy.

1 INTRODUCTION

Results from the optimal control design for linear systems with quadratic performance index (LQ problem) form the basis of modern control. Many systems are linear to begin with, while many nonlinear systems may be considered as linear when being operated near an equilibrium.

State-feedback LQ regulator (LQR) drives states $x(t)$ of a dynamic system from arbitrary initial conditions to zero for a zero setpoint. Asymptotic stability and required performance are provided by a proper choice of weighting matrices in quadratic performance index. Important modification is the state-feedback output regulator which drives to zero the output $y(t)$ [1]. If the controlled plant is observable, the output regulator problem can be reduced to the state regulator one. The main prerequisite for using LQR is availability of all plant states for feedback. In most real situations only some of them are measured as outputs; the design equations for the output feedback LQ regulator are more complicated than those for the state feedback one [1,3,6,7]. On the other hand, output-feedback design allows using compensators with any desired control system structure. One of the most important problems in control is making a system output to track a reference input signal.

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Solution to the general LQ tracking problem is not straightforward even using full state feedback; optimal solution is not causal but contains a feedforward term generated by a backward differential equation [1]. More suitable approaches for practical problems in [3] are based on using output feedback. For a constant setpoint the regulator can be converted into a tracker by adding additional feedforward terms. However, if the reference is not constant, feedforward terms generally contain also its derivatives. A powerful tracker design technique that automatically yields the precompensator required to guarantee proper tracking for a large class of command inputs is the command generator tracker (CGT) based on incorporating a model of the reference dynamics into the control system [3].

2 PRELIMINARIES

2.1 Problem formulation

Consider the linear state space model of the controlled plant

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x(t)$ is state vector, $u(t)$, $y(t)$ are control input and measured output of a single input-single output system, respectively; *A*, *B*, *C* are matrices of compatible dimensions. The performance output $z(t)$ (generally not equal to the measured output $y(t)$)

$$z = Hx \tag{2}$$

is required to track the reference input

$$r^{(d)} + a_1 r^{(d-1)} + \ldots + a_d r = 0 \tag{3}$$

In case of quadratic optimal tracking, the control $u(t)$ is to be found such that the tracking error

$$e(t) = r(t) - z(t) \tag{4}$$

asymptotically approaches zero.

2.2 CGT: continuous-time version (recalled)

A CGT design [3] is based on the internal model principle; according to it for proper asymptotic tracking the plant must model non-asymptotically stable modes of the reference signal. If it is not the case, the plant model has to be augmented to include those modes of the reference which are not at the same time modes of the plant [2]. Using CGT, a zero steady-state tracking error is achievable for a large class of command inputs describable by linear differential equations with constant coefficients

$$r^{(d)} + a_1 r^{(d-1)} + \ldots + a_d r = 0 \tag{5}$$

The corresponding characteristic polynomial

$$\Delta(s) = s^d + a_1 s^{d-1} + \ldots + a_d \tag{6}$$

expressed in the controllability canonical form is called the command generator system (e.g. for $d=3$)

$$\dot{\rho} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \rho = F\rho \,; \qquad r = [1 \quad 0 \quad 0]\rho \tag{7}$$

By suitably augmenting the original plant model, the tracking problem can be converted into the regulator problem with the error regulated to zero. Denote $\frac{d}{dt} \hat{=} s$, then (4) may be written as

$$\Delta(s) r = 0 \tag{8}$$

The tracking error is

$$e - r - z = r - Hx \tag{9}$$

Its dynamics is given by combining (5) and (9)

$$\Delta(s)e = \Delta(s)r - \Delta(s)Hx = -H\xi \tag{10}$$

where ξ is the state of the augmented system

$$\xi = \Delta(s)x = x^{(d)} + a_1 x^{(d-1)} + \ldots + a_d x \tag{11}$$

In a concise form, (9) can be written as follows

$$\dot{\varepsilon} = F\varepsilon + \begin{bmatrix} 0 \\ -H \end{bmatrix}\xi \tag{12}$$

where $\varepsilon = [e \;\; \dot{e} \;\; \ldots \;\; e^{(d-1)}]^T$. To determine dynamics of ξ , (5) is applied to (1) , hence

$$\dot{\xi} = A\xi + B\mu \tag{13}$$

where

$$\mu = \Delta(s)u = u^{(d)} + a_1 u^{(d-1)} + \ldots + a_d u \tag{14}$$

is the modified control input. Joining (11) and (12) we obtain the augmented state model

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\xi} \end{bmatrix} = \left[\begin{array}{c|c} F & \begin{array}{c} 0 \\ -H \end{array} \\ \hline 0 & A \end{array}\right] \begin{bmatrix} \varepsilon \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix}\mu \tag{15}$$

When the modified system state converges to zero, the tracking error vanishes. Thus, the optimal tracking problem has been transformed into the output LQ regulator design with the modified performance index

$$J = \int_0^\infty (\upsilon^T Q \upsilon + \mu^T R \mu)dt \tag{16}$$

where the output of the augmented model is

$$\upsilon = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} \varepsilon \\ \xi \end{bmatrix} \tag{17}$$

For appropriately selected weighting matrices *Q* and *R*, the resulting optimal control is

$$\mu = -\begin{bmatrix} K_\varepsilon & K_y \end{bmatrix}\begin{bmatrix} \varepsilon \\ C\xi \end{bmatrix} \tag{18}$$

or

$$\Delta(s)u = -K_\varepsilon \Delta(s)e - K_y C\Delta(s)x \tag{19}$$

The compensator is implemented using the transfer function derived from (18)

$$\frac{u + K_y y}{e} = -\frac{K_1 s^{d-1} + \ldots + K_{d-1}s + K_d}{s^d + a_1 s^{d-1} + \ldots + a_d} \tag{20}$$

2.3 Tracking with Disturbance Rejection

The CGT may also be used in disturbance rejection problem [3]. Let the disturbance *d(t)* satisfy the differential equation (e.g. for constant disturbance $q=1$, for sinusoidal disturbance $q=2$).

$$d^{(q)} + p_1 r^{(q-1)} + \ldots + p_d d = 0 \tag{21}$$

Rewrite (8) to emphasize that we deal with the characteristic equation of the reference as follows

$$\Delta_r(s)r = 0 \tag{22}$$

and define $$\Delta(s) = \Delta_d(s)\Delta_r(s) \tag{23}$$

If we apply the CGT technique with $\Delta(s)$ according to (23), the resulting compensator will guarantee tracking of *r(t)* by the performance output (2) in presence of the disturbance *d(t)* (21). If $\Delta_d(s)$ and $\Delta_r(s)$ have common factors, it is sufficient to generate (23) using their least common multiple.

3 DISCRETE-TIME CGT DESIGN

The discrete-time CGT (DCGT) is designed using a similar procedure with discretized command generator and plant model. The discrete-time command generator matrix F_d is obtained by factorizing the characteristic polynomial (5)

$$\Delta(s) = \prod_{i=1}^{n}(s - s_i) \tag{24}$$

and using the substitution $z_i = e^{s_i T}$ (T is the sampling period) to obtain

$$\Delta(z) = \prod_{i=1}^{n}(z - z_i) = z^d + a_1 z^{d-1} + \ldots + a_d \tag{25}$$

where F_d - the discrete counterpart to (7) is obtained from (25). The discrete plant model is obtained using the common conversion formulas

$$A_d = e^{AT};\ B_d = (e^{AT} - I)A^{-1}B \tag{26}$$

whereby matrices C and H remain unchanged. The augmented performance index becomes

$$J_d = \sum_{k=0}^{\infty} \upsilon_k^T Q_d \upsilon_k + \mu_k^T R_d \mu_k \tag{27}$$

where $Q_d = Q.T$, $R_d = R.T$. The transfer function for implementation of the compensator is

$$\frac{u_k + K_y y_k}{e_k} = -\frac{K_1 z^{d-1} + \ldots + K_{d-1} z + K_d}{z^d + a_1 z^{d-1} + \ldots + a_d} \tag{28}$$

The design procedure is exhaustively explained on the following case study.

4 CASE STUDY

Consider a DC motor [5] with the state space model as follows

$$\frac{d}{dt}\begin{bmatrix} i \\ \omega \end{bmatrix} = \begin{bmatrix} -333.3 & -92.3333 \\ 2000 & -728 \end{bmatrix}\begin{bmatrix} i_a \\ \omega \end{bmatrix} + \begin{bmatrix} 1666.7 \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ -20000 \end{bmatrix} M_z \tag{29}$$

where the state variables are ω (speed) and i_a (armature current); input variables are the armature voltage u_k (control) and the load M_z (disturbance); output variables are ω and i_a hence $C = I_{2x2}$.

The task is to design a control *u(t)* guaranteeing that the DC motor speed ω follows the reference ω_{ref} with trapezoidal profile and has reduced sensitivity to load step changes. According to the task formulation, the performance output is

$$\omega = [0 \quad 1]\begin{bmatrix} i \\ \omega \end{bmatrix} \quad \text{where} \quad H = [0 \ \ 1] \tag{30}$$

Trapezoidal profile of the reference speed is made up of ramp and step signals; differential equation of a ramp with the slope ν_0 is

$$\ddot{r} = 0\ ,\ r(0) = 0,\ \dot{r}(0) = \nu_0 \tag{31}$$

The corresponding characteristic equation $\Delta(s) = s^2 = 0$ has a double root $s_{1,2} = 0$.

A step signal r_0 is described by a differential equation

$$\dot{r} = 0, \quad r(0) = r_0 \tag{32}$$

and the corresponding characteristic equation $\Delta(s) = s = 0$ has one root $s = 0$.

Hence, by guaranteeing tracking of a ramp signal, we implicitly guarantee tracking of a step reference and also rejection of a step and/or ramp disturbances. To obtain the discrete ramp-type command generator system, we substitute for $z_i = e^{s_i T}$, $s_{1,2} = 0$ in (25) to obtain

$$\Delta(z) = (z-1)^2 = z^2 - 2z + 1 \tag{33}$$

Hence the discrete command generator matrix is

$$F_d = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \tag{34}$$

The resulting augmented plant model (Fig. 1)

$$\begin{bmatrix} e_{k+1} \\ e_{k+2} \\ i_{k+3} \\ \omega_{k+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & a_{11d} & a_{12d} \\ 0 & 0 & a_{21d} & a_{22d} \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \\ i_{k+2} \\ \omega_{k+2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_{1d} \\ b_{2d} \end{bmatrix} \mu_k \tag{35}$$

where A_d and B_d are obtained according to (26).

$$A_d = \begin{bmatrix} a_{11d} & a_{12d} \\ a_{21d} & a_{22d} \end{bmatrix} \quad \text{and} \quad B_d = \begin{bmatrix} b_{1d} \\ b_{2d} \end{bmatrix} \tag{36}$$

The augmented output vector is $\upsilon_k = \begin{bmatrix} e_k & e_{k+1} & i_{k+2} & \omega_{k+2} \end{bmatrix}^T$; the control vector is $\mu_k = u_{k+2}$. By creating the augmented plant model, the tracking problem has been converted to state feedback output LQ regulator problem. Optimal control of the augmented system is

$$\mu_k = u_{k+2} = -K_d \upsilon_k = -k_{1d} e_k - k_{2d} e_{k+1} - k_{3d} i_{k+2} - k_{4d} \omega_{k+2} \tag{37}$$

Control of the original system is obtained by shifting μ_k by two steps

$$u_k = \mu_{k-2} = -k_{1d} e_{k-2} - k_{2d} e_{k-1} - k_{3d} i_k - k_{4d} \omega_k \tag{38}$$

Design results obtained using standard state-feedback output LQ controller design procedure with the weighting matrices in (38)

$$Q_d = \begin{bmatrix} 10^4 T & 0 & 0 & 0 \\ 0 & 10^4 T & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \end{bmatrix} \quad R_d = 0.01T \tag{39}$$

for various sampling periods are summarized in Tab. 1.

Tab. 1 Summarized design results for various sampling periods

Sampling period	k_{1d}	k_{2d}	k_{3d}	k_{4d}
T = 0.01s	*0.254316*	*- 0.381474*	*- 0.002066*	*0.254029*
T = 0.015s	*0.256407*	*- 0.384610*	*- 0.000123*	*0.256458*
T = 0.02s	*0.256407*	*- 0.384610*	*- 0.000016*	*0.256406*

4 CONCLUSIONS

The CGT design is simple and direct to apply. The resulting compensator includes both feedback and feedforward terms so that both the closed-loop poles and zeros may be affected by varying the gain matrix *K*. The method is applicable if the original plant is reachable and the loop transfer function $H(z) = H(zI - A_d)^{-1} B_d$ has no zeros at the roots of $\Delta(z) = 0$.

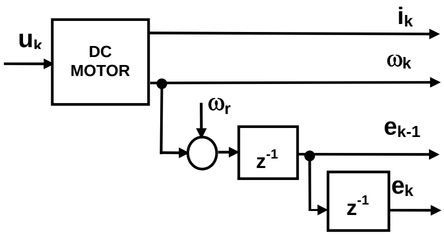

Fig.1 Explicative block scheme of the augmented discrete-time DC motor model

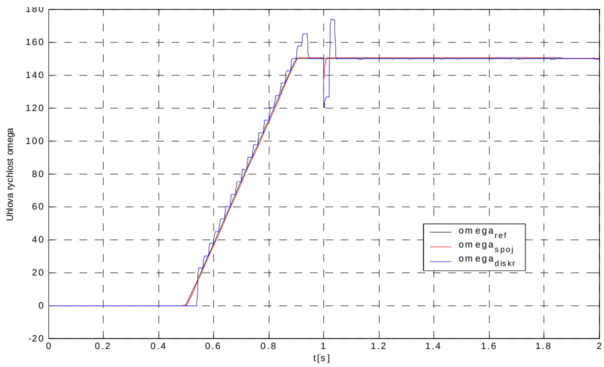

Fig. 2 Comparison of CGT and DCGT for *T=0.02s*.

Tracked reference is the trapezoidal-form speed profile; rejected disturbance is a load step change M_z*=1.5Nm* occurred in *1s*.

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