Transactions of the VŠB – Technical University of Ostrava, Mechanical Series
No. 2, 2008, vol. LIV,
article No. 1618

Dariusz JANECKI[*], Leszek CEDRO[**]

SAMPLED SIGNALS DIFFERENTIATION USING A FREQUENCY CORRECTED DIFFERENCE QUOTIENT THERMS

DERIVOVÁNÍ VZORKOVANÝCH SIGNÁLŮ POUŽITÍM FREKVENČNĚ KORIGOVANÉ DIFERENCE

Abstract

The work discusses the methodology of designing finite impulse response differential filters. The filters require combining a difference quotient term of a proper order with a low-pass correction term responsible for the correction of the difference quotient term characteristic and elimination of measurement and quantization noise. The filters well approximate the characteristic of ideal differentiating elements in the low frequency range. All calculations were executed in “Mathematica”.

Abstrakt

Práce popisuje metodologii návrhu číslicových filtrů s konečnou odezvou pro derivaci. Filtry požadují propojení rozdílového členu vlastního řádu s dolnopropustným korekčním členem plnící funkci eliminace měřicího a kvantizačního šumu. Filtry dobře aproximují charakteristiku ideálního diferenčního členu v nízkofrekvenčním rozsahu.Všechny výpočty byly provedeny v programu “Mathematica”.

1 INTRODUCTION

The method of identification based on the inverse model of a plant as well as its generalization require information on derivatives of output and input signals with respect to time. The required order for these derivatives depends on the order of the differential equations describing the object. In practice, signal derivatives are seldom measured directly; they are usually determined on the basis of registered signals. Nowadays computer-based techniques are commonly used for identifying and controlling systems. This means we no longer deal with continous-time signals; we use their samples obtained at regular intervals called sampling periods. Signal derivatives can be determined basing on the signal samples only.

2 PROBLEM FORMULATION

The problem can be formulated as follows: for a given continous-time signal $x(t)$, using the signal samples $x_n = x(n\Delta)$, $n = \ldots -2,-1,0,1,2,\ldots$, estimate the values of the derivatives, $\dot{x}(n\Delta)$, $\ddot{x}(n\Delta) \ldots x^{(k)}(n\Delta)$, where Δ is the sampling period and k is the highest required order for the derivative. The system for differentiating continous-time signals can be written as:

$$y(t) = D^k x(t)\,, \qquad (1)$$

[*] Assoc. Prof., Kielce University of Technology, Faculty of Mechatronics and Machinery Design, Division of Computer Science and Robotics, Al. Tysiąclecia PP 7, 25-314 Kielce, Poland, tel/fax. +48/41/3424504, email djanecki@tu.kielce.pl
[**] DrSc, e-mail lcedro@tu.kielce.pl

where D is the differentiation operator. The frequency characteristics of the differential term can be calculated by applying transfer function $H_k(\omega) = (j\omega)^k$ in the continuous-time domain or transfer function :

$$H_k(\Omega) = \left(\frac{j\Omega}{\Delta}\right)^k, \ |\Omega| \le \pi \tag{2}$$

in the discrete-time domain, where $\Omega = \Delta\omega$ is the normalized angular frequency of the signal.

The aim of the study is to design a discrete system (algorithm) with a transfer function approximating transfer function (2). Since measurement signals are usually disturbed by broadband noise resulting, for instance, from the process of signal quantization, assume that the characteristic of a differential filter will be approximated in lower frequencies only. It is required, then, that the differential systems to be developed approximate the following transfer function:

$$H_k(\Omega) = \begin{cases} \left(\dfrac{j\Omega}{\Delta}\right)^k & \text{for} \quad \Omega \le \Omega_g \\ 0 & \text{for} \quad \Omega > \Omega_g . \end{cases} \tag{3}$$

Transfer function $H_k(\Omega)$ can be treated as a series connection of an ideal differential term and an ideal low-pass filter with cut off frequency Ω_g .

3 SIMPLE METHODS OF SIGNAL DIFFERENTIATION

The most natural method of determining a signal derivative seems the following approximation:

$$x^{(1)}(n\Delta) \cong \frac{x(n\Delta) - x(n\Delta - \Delta)}{\Delta} . \tag{4}$$

As can be expected, the shorter the sampling period, the better the estimation results. Equation (5) will have the following form:

$$x^{(1)}(n\Delta) \cong \frac{1-q^{-1}}{\Delta} x(n\Delta) \overset{df}{=} \nabla^1 x(n\Delta) , \tag{5}$$

where q^{-1} is the backward signal shift operator by one sampling period, that is

$$q^{-1}x(n\Delta) = x(n\Delta - \Delta) . \tag{6}$$

Expression $\nabla^1 = (1-q^{-1})/\Delta$ is called a difference quotient of the first order. The difference quotient defined by Eq. (5), however, results in a phase shift of the output signal, $x^{(1)}(n\Delta)$. To avoid this, Eq. (5) is replaced by a modified difference quotient having the following form: $\nabla^1 = (q - q^{-1})/2\Delta$. Generally, the operation of the k -th order differentiation can be realized using a difference quotient of the k-th order:

$$\nabla^k = \begin{cases} \dfrac{(q-q^{-1})(q-1)^l(1-q^{-1})^l}{2\Delta^k} & \text{for} \quad k = 2l+1 \\ \dfrac{(q-1)^l(1-q^{-1})^l}{\Delta^k} & \text{for} \quad k = 2l. \end{cases} \tag{7}$$

4 DIFFERENTIAL FILTERS

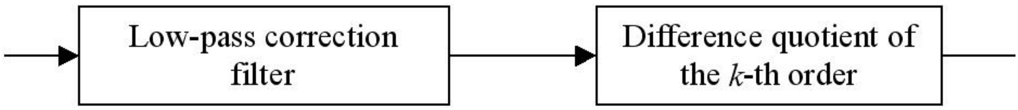


Fig. 1 Block diagram of the differential filter

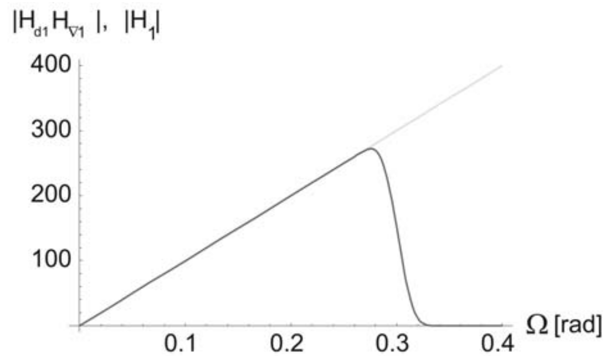
Let us assume that the differential filter of the *k*-th order is a series connection of a low-pass filter with boundary frequency Ω_g and a difference quotient of the *k*-th order (Fig.1). The low-pass filter will be responsible firstly for reducing the signal spectrum and secondly for correcting the characteristics of the difference quotient in the range of low frequencies. Thus, the filter will be called a low-pass correction filter. The desired transfer function of the low-pass filter is:

$$H_{\mathrm{kor}\,k}(\Omega) = \begin{cases} H_k(\Omega)/H_{\Delta k}(\Omega) & \text{for} \quad \Omega \le \Omega_g \\ 0 & \text{for} \quad \Omega > \Omega_g, \end{cases} \tag{8}$$

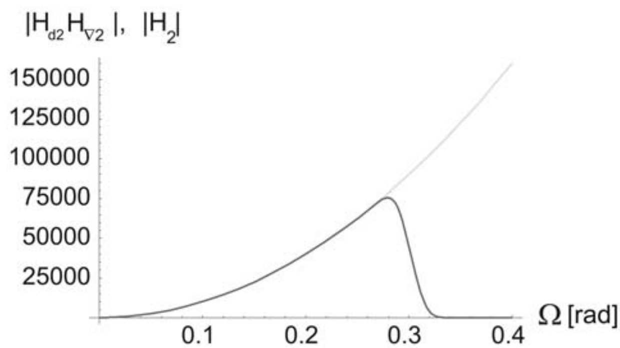
where $H_{\Delta k}(\Omega)$ is the transfer function of the *k*-th order difference quotient: $H_{\Delta k}(\Omega) = \nabla^k \Big|_{q=e^{j\Omega}}$

As a result, the transfer function of the series connection of the difference quotient and the low-pass filter in the range of low frequencies will approximate to the transfer function of an ideal differential filter (3).

$k = 1$



$k = 2$



$k = 3$

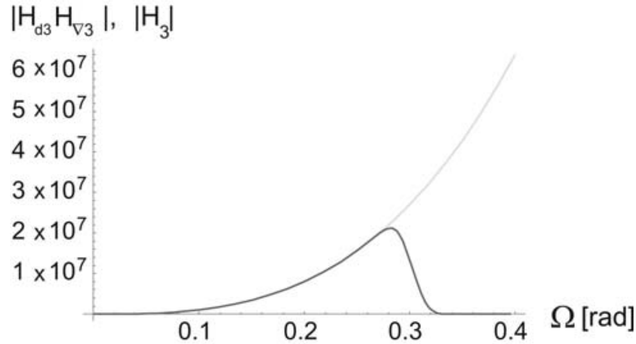


Fig. 2 The frequency characteristic of developed differential filter for different filter order $k = 1, 2, 3$.

It can be show that the transfer function of the low-pass filter for $\Omega \le \Omega_g$ is equal to

$$H_{\mathrm{kor}k}(\Omega) = H_k(\Omega)/H_{\nabla k}(\Omega) = \begin{cases} \Omega/\sin\Omega & k=1 \\ \Omega^2/2(1-\cos\Omega) & k=2 \\ \Omega^3/(-2\sin\Omega+\sin(2\Omega)) & k=3. \end{cases} \tag{9}$$

The filter impulse response is the inverse Fourier transform of its frequency characteristic, thus:

$$h_{\mathrm{kor}k}(n) = \frac{1}{2\pi}\int_{-\Omega_g}^{\Omega_g} H_{\mathrm{kor}k}(\Omega)e^{j\Omega n}d\Omega\,. \tag{10}$$

Unfortunately, integral (10) cannot be expressed by means of the analytic functions. It needs to be determined using some approximation. By expanding function $H_{\mathrm{kor}k}(\Omega)$ into a Taylor series around the value $\Omega = 0$, we obtain:

$$H_{\mathrm{kor}k}(\Omega) = \begin{cases} 1+\dfrac{\Omega^2}{6}+O(\Omega^4) & k=1 \\ 1+\dfrac{\Omega^2}{12}+O(\Omega^4) & k=2 \\ 1+\dfrac{\Omega^2}{4}+O(\Omega^4) & k=3. \end{cases} \tag{11}$$

The four-term approximation of the expansion appears to be fairly sufficient. The inverse Fourier transform of the function obtained by rejecting the terms of the higher orders is equal to:

$$h_{\mathrm{kor}k}(n) = \begin{cases} \dfrac{12}{6n^3\pi}n\Omega_g\cos(n\Omega_g)+(6n^2+\Omega^2n^2-2)\sin(n\Omega_g) & k=1 \\ \dfrac{1}{12n^3\pi}(2n\Omega_g\cos(n\Omega_g)+(12n^2+n^2\Omega_g^2-2)\sin(n\Omega_g)) & k=2 \\ \dfrac{1}{4n^3\pi}(2n\Omega_g\cos(n\Omega_g)+(4n^2+n^2\Omega_g^2-2)\sin(n\Omega_g)) & k=3. \end{cases} \tag{12}$$

We will implement the correction filter using FIR filter design method. Assume that the impulse response of the low-pass correction filter is:

$$h_{dk}(n) = \frac{1}{\chi_k} h_{\mathrm{kor}\,k}(n) W_{\mathrm{Harris}}(n) , \tag{13}$$

where $W_{\mathrm{Harris}}(n)$ is Harris window [2] described by the following equation:

$$W_{\mathrm{Harris}}(n) = \begin{cases} 0{,}36 + 0{,}49\cos(\pi n/M) + 0{,}14\cos(2n\pi/M) + 0{,}01\cos(3n\pi/M) & \text{for} \quad |n| \le M \\ 0 \quad \text{otherwise,} & \end{cases} \tag{14}$$

where M is a suitable chosen positive integer.

The parameter χ_k should be selected in such a way that the *k*-th derivative of the characteristic of the filter being designed at point $\Omega = 0$ be the same as that of the ideal differential therm, thus:

$$\frac{\partial^k}{\partial \Omega^k} H_{dk}(\Omega) H_{\nabla k}(\Omega)\,|_{\Omega=0} = \frac{k!\, j^k}{\Delta^k} . \tag{15}$$

The frequency characteristics of the developed differential filter for different filter order $k = 1, 2, 3$ and $\Omega_g = 0.3$, $M = 300$ are shown on the figure 2. For comparision, the characteristics of the ideal differentiating term are show on this figure too.

5 RESULTS OF DIFFERENTIATION BY MEANS OF DEVELOPED QUOTIENTS AND DIFFERENTIAL FILTERS

We shall now examine, by way of simulation, the quality of derivative determination by means of the developed differential filters. Let us consider an operation of differentiation of the second order of a sinusoidal signal

$$x(t) = \sin \omega t, \tag{16}$$

with $\omega = 10\pi$. Assume that the signal is sampled with a sampling period of $\Delta = 0.001\,[s]$ and converted from an analogue to a digital form by using an analogue to digital converter with a range of $[-1, 1]$ and $m = 16$ -bit processing.

Figures 3 and 4 compares the signals obtained by differentiation using difference quotients and the developed differential filters with those obtained by ideal differentiation.

Fig. 3 Relative signal difference $\nabla^2 x_{\mathrm{kw}} - D^2 x$

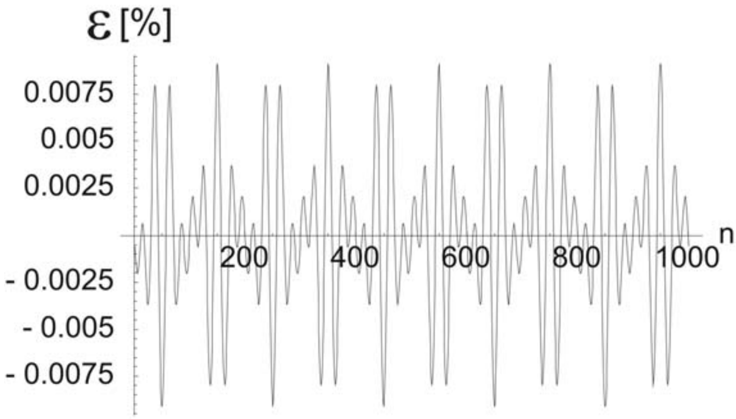


Fig. 4 Relative error after employing the developed differential filter of the second order

As can be seen, the influence of the quantization error for the filters being analyzed can be reduced substantially. In the case of a second-order filter, the error is not more than 0.008%. It should be noted that the error of differentiation of quantized signals is greatly dependent on the signal frequency. The lower the frequency, the bigger the error.

6 CONCLUDING REMARKS

The proper selection of parameters M and Ω_g of the differential filters is of great importance. The filter window length, $2M+1$, affects the accuracy of approximation of the ideal filter characteristics. The higher the value of M, the better the approximation accuracy. Increasing the window length, however, will lengthen the calculation time and slightly shorten the filtered signals. The specially designed differential filters make it possible to select appropriate signal derivatives, with errors being much smaller than those encountered in simple methods of differentiation, which is of great significance in the process of identification.

REFERENCES

[1] LYNN PAUL. A., *Digital Signal Processing with Computer Application*, 1989, John Wiley & Sons.

[2] LYONS R.G. *An introduction to digital signal processing* (in Polish), WKiŁ, Warsow 1999.

[3] SODERSTROM T. & STOICA P. *System Identification*. PHI (UK), 1994.

[4] WOLFRAM S.: *The Mathematica Book*, 5th ed., Wolfram Media 2003.

Reviewers:

prof. Ing. Roman Prokop, CSc., Tomas Bata University in Zlín

prof. Ing. Antonín Víteček, CSc., VŠB-Technical University of Ostrava