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BEAM ON ELASTIC FOUNDATION WITH LONGITUDINAL CHANGES
(SOLUTION VIA SBRA METHOD)

NOSNÍK NA PRUŽNÉM PODKLADU S PODÉLNÝMI ZMĚNAMI
(ŘEŠENÍ METODOU SBRA)

Abstract

This paper is focused on the solution of simple beam continually supported by elastic (Winkler's) foundation. The foundation contains possible longitudinal changes of stiffness, which can be caused by material of foundation. For the calculation of displacements and bending stresses are derived and applied some analytical procedures (approximate solution in the form of polynomial function) and probabilistic approaches (SBRA - Simulation-Based Reliability Assessment method, Monte Carlo Simulation Method, AntHill software). Probabilistic approach includes influences of real variability of load, shape and material of the beam, and also real variability of modulus of the foundation. Probabilistic approach is used for the reliability expertise of the beam and calculation of safety. Probabilistic expertise of the beam describes better the reality than the clasical deterministic expertise.

Abstrakt

Článek se zaměřuje na řešení jednoduchého nosníku spojitě podloženého elastickým (Winklerovým) podložím. Podloží obsahuje možné podélné změny tuhosti, které mohou být způsobeny materiálem podloží. Pro výpočty průhybů, ohybových napětí jsou odvozeny a aplikovány analytické postupy (přibližné řešení ve tvaru polynomické funkce) a pravděpodobnostní přístupy (metoda SBRA - Simulation-Based Reliability Assessment, simulace Monte Carlo, AntHill software). Pravděpodobnostní přístup zahrnuje vlivy reálné variabilityzatížení, tvaru a materiálu nosníku a také reálnou variabilitu modulu podloží. Pravděpodobnostní přístup je použitý pro pravděpodobnostní posudek spolehlivosti nosníku a výpočtu bezpečnosti. Pravděpodobnostní posudek nosníku popisuje lépe skutečnost než klasický deterministický posudek spolehlivosti.

1 INTRODUCTION

The analysis of bending of beams on an elastic foundation is developed on the assumption that the strains are small and the reaction forces $\mathrm{q_R} = \mathrm{q_R}(x)$ [Nm^{-1}] in the foundation are proportional at every point to the deflection $v = v(x)$ [m] of the beam at that point, etc. (first proposed by E. Winkler, Prague 1867), see also Fig.1. External loads on the beam also evoke bending moment M_o [Nm], axial (normal) force N [N] and shearing force T [N], see Fig.1. The general problem is described by ordinary differential equation:

$$\frac{d^4v}{dx^4} - \frac{N}{EJ_{ZT}}\frac{d^2v}{dx^2} + \frac{\beta}{GA}\frac{d^2\mathrm{q_R}}{dx^2} + \frac{\mathrm{q_R}}{EJ_{ZT}} = \frac{1}{EJ_{ZT}}\left(\mathrm{q} - \frac{dm}{dx}\right) + \frac{\beta}{GA}\frac{d^2\mathrm{q}}{dx^2} - \frac{\alpha_\mathrm{t}}{\mathrm{h}}\frac{d^2(\mathrm{t_2} - \mathrm{t_1})}{dx^2}\ , \tag{1}$$

where: E [Pa] is modulus of elasticity in tension of the beam, $J_{\mathrm{ZT}} = \int_A y^2 dA$ [m^4] is the major principal second moment of area A [m^2] of the beam cross-section, β [1] is shear deflection constant of the beam, G [Pa] is modulus of elasticity in torsion of the beam, q [Nm^{-1}] is distributed load (inten-

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sity of force), m [N] is distributed couple (intensity of moment), α_t [deg^{-1}] is coefficient of thermal expansion of the beam, h [m] is depth of the beam and $t_2 - t_1$ [deg] is transversal temperature increasing in the beam. For more information about the derivation of eq. (1), see reference [1].

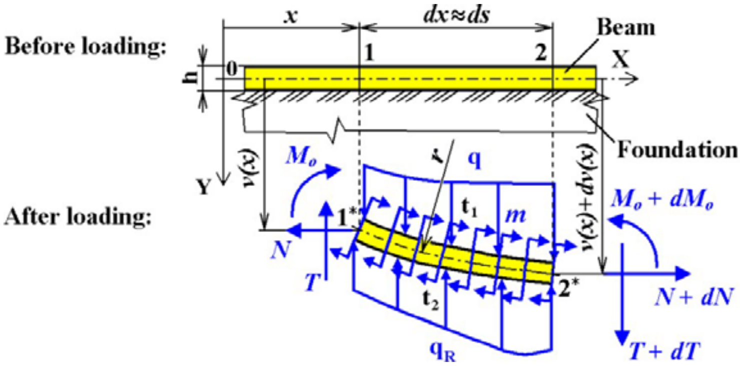


Fig. 1 Element of a Beam on Elastic Foundation

In the most situations, the influences of force, shear force, distributed load, distributed couple and temperature can be neglected (or the beam is not exposed to them). Hence, from eq. (1) follows:

$$\frac{d^4v}{dx^4} + \frac{q_R}{EJ_{ZT}} = 0 \ , \tag{2}$$

where (from the Winkler's theory, see [1] or [5]), is evident that::

$$q_R = v \times k(x) = v \times b \times K(x) \ , \tag{3}$$

and where functions: $k(x)$ [Pa] is stiffness of the foundation and $K(x)$ [Nm^{-3}] is modulus of the foundation which can be expressed as functions of variable x [m] (i.e. longitudinal changes in the foundation caused by properties of foundation) and b [m] is width of the beam. Hence, eq. (2) can be written in the form:

$$\frac{d^4v}{dx^4} + \frac{b\,K(x)}{EJ_{ZT}} v = 0 \ . \tag{4}$$

2 EXAMPLE OF GENERAL SOLUTION (DERIVATION)

Let us consider the straight short beam on elastic nonlinear foundation, see Fig.2. The beam of length L [m] with free ends is exposed to one vertical force F [N]. Modulus of the foundation is given by linear function:

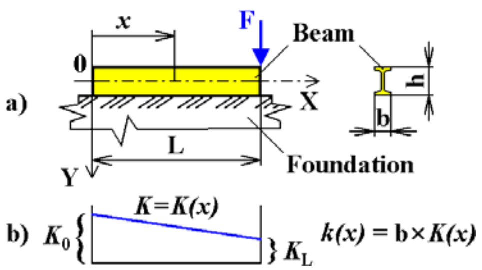


Fig. 2 Solved Example of the Beam on Elastic Foundation with Longitudinal Changes

$$K(x) = K_0 + \frac{K_L - K_0}{L} x = K_0 + K_1 x \ . \tag{5}$$

Hence, in this case, the differential eq. (4) can be written in the form:

$$\frac{d^4v}{dx^4} + \frac{(K_0 + K_1 x) b}{EJ_{ZT}} v = 0 \ . \tag{6}$$

The exact solution of equation (6) cannot be found. However, the approximate solution can be found in the form of polynomial function of 6$^{th.}$ order:

$$v = v(x) \approx b_0 + \sum_{i=1}^{6} \frac{b_i}{i} x^i \ , \tag{7}$$

where: b_0 [m], b_1 [1], $\cdots$, b_6 [m^{-5}] are unknown constants.

Equation (7) must satisfy the basic boundary conditions (at the point $x = 0$: $M_o(x=0) = 0$, $T(x=0) = 0$ and at the point $x = \mathrm{L}$: $M_o(x=\mathrm{L}) = 0$, $T(x=\mathrm{L}) = \mathrm{F}$). Force equation of equilibrium must be also satisfied (i.e. equation: $\int_0^{\mathrm{L}} q_{\mathrm{R}}\, dx = \int_0^{\mathrm{L}} \mathrm{b}\,(K_0 + K_1 x)\left(b_0 + \sum_{i=1}^{6} \frac{b_i}{i} x^i\right) dx = \int_0^{\mathrm{L}} (k_0 + k_1 x)\left(b_0 + \sum_{i=1}^{6} \frac{b_i}{i} x^i\right) dx = \mathrm{F}$), where: $k_0 = K_0 \mathrm{b}$ [Pa] is the stiffness in the foundation at the point $x = 0$ and $k_1 = K_1 \mathrm{b} = \frac{K_{\mathrm{L}} - K_0}{\mathrm{L}} \mathrm{b}$ [Nm^{-3}] is the slope of a given linear function $k(x)$, see also Fig.2.

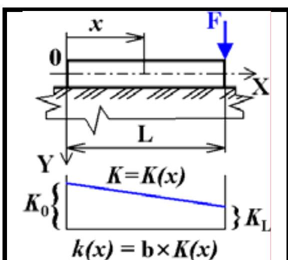
From the above five conditions can be expressed constants b_0 and $b_2, \cdots, b_5$ as functions of two constants b_1 and b_6. The last two constants (i.e. b_1 and b_6) can be derived via variational principles or via satisfaction of differential equation (6) at chosen points. For more details about it see [2]. The auxiliary constants $\overline{\mathrm{A}}$ /N^{2}m^{-1}/, $\overline{\mathrm{B}}$ /N^2/, $\overline{\mathrm{C}}$ /N^{-3}m^{-2}/ and remaining polynomial constants (b_1 and b_6) are derived in the Tab.1.

Tab. 1 Solved Example (Auxiliary Constants and Used Polynomial Constants)

$k_0 = K_0 \mathrm{b}$	$k_1 = K_1 \mathrm{b} = \frac{K_{\mathrm{L}} - K_0}{\mathrm{L}} \mathrm{b}$	$b_6 = -2\overline{\mathrm{C}}\left[840\,\overline{\mathrm{B}}\,k_1 + (21 k_0 + 11 k_1 \mathrm{L})\overline{\mathrm{A}}\right]$	$k_0 = K_0 \mathrm{b}$
$\overline{\mathrm{C}} = \frac{\mathrm{F}}{15 E J_{ZT} \mathrm{L}^2 \left(560 E J_{ZT}\left[6 k_0 (k_0 + k_1 \mathrm{L}) + k_1^2 \mathrm{L}^2\right] + 3\overline{\mathrm{A}}(2k_0 + k_1 \mathrm{L})\mathrm{L}\right)}$			$K(x) = K_0 + K_1 x$ $\overline{\mathrm{A}} = k_0 (k_0 + k_1 \mathrm{L}) \mathrm{L}^3$
$b_1 = \frac{\overline{\mathrm{C}}}{6}\left[604800\,\overline{\mathrm{B}}\,E J_{ZT} - 180\left(41 k_0 k_1 \mathrm{L} + 9 k_1^2 \mathrm{L}^2 + 38 k_0^2\right) E J_{ZT} \mathrm{L}^4 + (3 k_0 + 2 k_1 \mathrm{L})\overline{\mathrm{A}}\,\mathrm{L}^5\right]$			
$\overline{\mathrm{B}} = E J_{ZT} (k_1 \mathrm{L} + 3 k_0)$		$k_1 = K_1 \mathrm{b} = \frac{K_{\mathrm{L}} - K_0}{\mathrm{L}} \mathrm{b}$	

The analytical results (i.e. functions of displacement v, slope $\frac{dv}{dx}$ [rad], bending moment M_o and shearing force T of the beam) are written in the Tab.2.

Tab. 2 Solved Example (Results of the Beam on Elastic Foundation with Longitudinal changes).



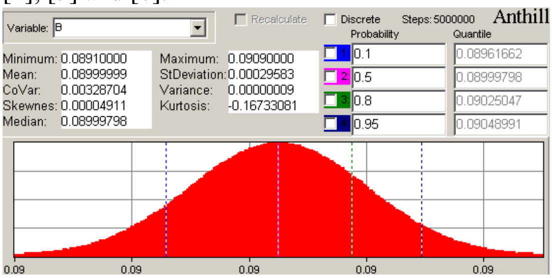
$M_o = -E J_{ZT} \frac{d^2 v}{dx^2} \approx \frac{\mathrm{F} x^2 (x - \mathrm{L})}{\mathrm{L}^2} - 5 E J_{ZT} x^2 (x - \mathrm{L})^2 b_6$		$x \in (0; \mathrm{L})$
$T = -E J_{ZT} \frac{d^3 v}{dx^3} \approx 10 E J_{ZT} x (3x\mathrm{L} - 2x^2 - \mathrm{L}^2) b_6 + \frac{\mathrm{F} x (3x - 2\mathrm{L})}{\mathrm{L}^2}$		
$\frac{dv}{dx} \approx b_1 + \frac{x^3 (10\mathrm{L}^2 - 15x\mathrm{L} + 6x^2)}{6} b_6 + \frac{\mathrm{F} x^3 (4\mathrm{L} - 3x)}{12 E J_{ZT} \mathrm{L}^2}$		See also Tab.1
$v \approx b_1 x + \frac{k_0 x^4 (5\mathrm{L}^2 - 6x\mathrm{L} + 2x^2) - 120 E J_{ZT} \mathrm{L}^2}{12 k_0} b_6 + \frac{\mathrm{F}\left[k_0 x^4 (5\mathrm{L} - 3x) - 120 E J_{ZT} \mathrm{L}\right]}{60 k_0 E J_{ZT} \mathrm{L}^2}$		

The accuracy of the derived results (Tab. 2) was also checked by ANSYS software (see [2]). However, the derived results fits very well for short beams (i.e. for the situations when the length of the beam $L \le 1$ m). For longer beams must be used higher approximation, i.e. function: $v = v(x) \approx b_0 + \sum_{i=1}^{n} \frac{b_i}{i} x^i$, where $n \ge 7$.

3 PROBABILITY ANALYSIS OF THE BEAM - inputs

Deterministic approach (i.e. all inputs are constant) is the elder but simple way how to get the solution of mechanical systems. However, the deterministic approach cannot truly include the real variability of all inputs.

But this example is solved via probabilistic approach (i.e. all inputs are given by bounded (truncated) histograms) which is the modern and new trend of the solution of mechanical systems, see [2], [5] and [6].



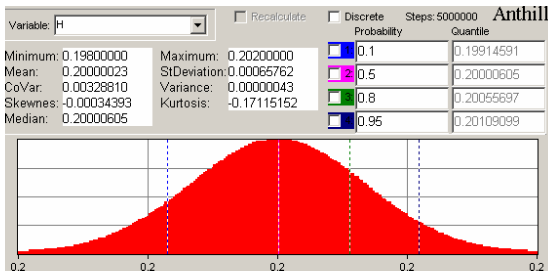
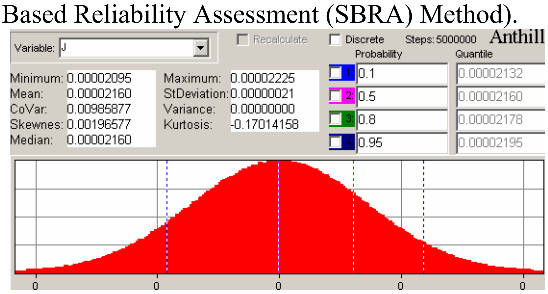


Fig. 3 Histogram of Input Parameter b [m] **Fig. 4** Histogram of Input Parameter h [m]

Probability analysis (see references [5] and [6]) of the presented beam (see Fig.2) includes influences of variability of "I" shape ($b = 0.09 \pm 9 \times 10^{-4}$ [m], $h = 0.2 \pm 2 \times 10^{-3}$ [m], $J_{ZT} = 2.16 \times 10^{-5} \pm 6.5 \times 10^{-7}$ [m^4]), variability of material: ($E = 1.8 \times 10^{11} \pm 9 \times 10^{9}$ [Pa], yield stress $R_e = 162.361 \times 10^{11}\,{}^{+77.587}_{-43.345}$ [MPa]), variability of load ($F = 157324.2\,{}^{+168773.1}_{-75524.2}$ [N]) and also the variability of modulus of the foundation: ($K_0 = 1.125 \times 10^{11} \pm 3.375 \times 10^{10}$ [Nm^{-3}], $K_L = 1.125 \times 10^{11} \pm 3.375 \times 10^{10}$ [Nm^{-3}]), see Fig.3 to 10 (i.e. inputs for AntHill software, Simulation-Based Reliability Assessment (SBRA) Method).



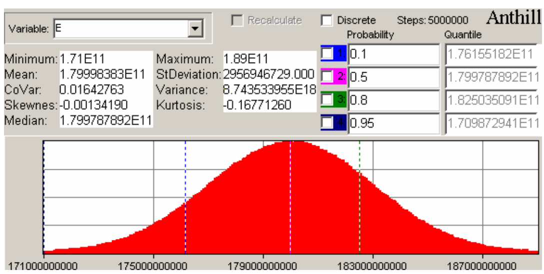
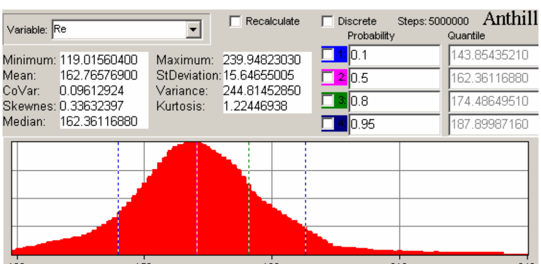


Fig. 5 Histogram of Input Parameter J_{ZT} [m^4] **Fig. 6** Histogram of Input Parameter E [Pa]



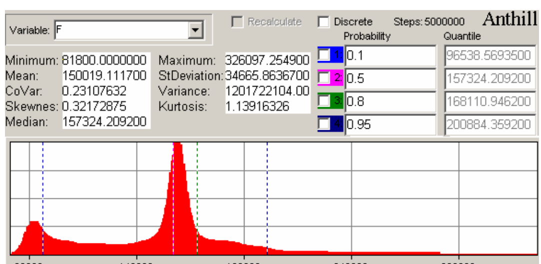


Fig. 7 Histogram of Input Parameter R_e [MPa] **Fig. 8** Histogram of Input Parameter F [N]

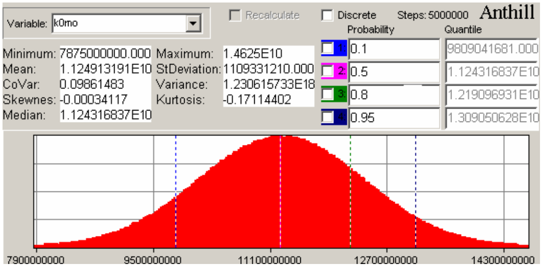


Fig. 9 Histogram of Input Parameter K_0 [Nm^{-3}]

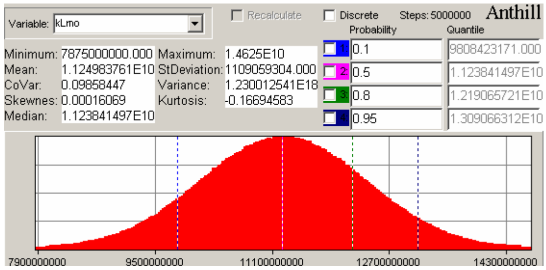


Fig. 10 Histogram of Input Parameter K_L [Nm^{-3}]

4 PROBABILITY ANALYSIS OF THE BEAM - OUTPUTS

The values of results parameters (i.e. stiffness of the foundation $k(x)$, displacement $v(x)$, maximal displacement $v_{\mathrm{MAX}} = v(x = \mathrm{L})$, bending stress $\sigma(x)$ and maximal bending stress $\sigma_{\mathrm{MAX}} = \sigma(x \approx 0.63) = \dfrac{|M_{o\,\mathrm{MAX}}|}{W_o} = \dfrac{|M_{o\,\mathrm{MAX}}|\,\mathrm{h}}{2\,J_{ZT}}$) were calculated for 5×10^6 simulations by Monte Carlo Method. Results are plotted by histograms in the following Figures 11 to 14 and Tab.3.

Hence, from the presented results is evident that maximal displacement is at the right end of the beam (i.e. at the point $x = \mathrm{L} = 0.9\ \mathrm{m}$) and maximal stress is at the point $x \approx 0.63\ \mathrm{m}$.

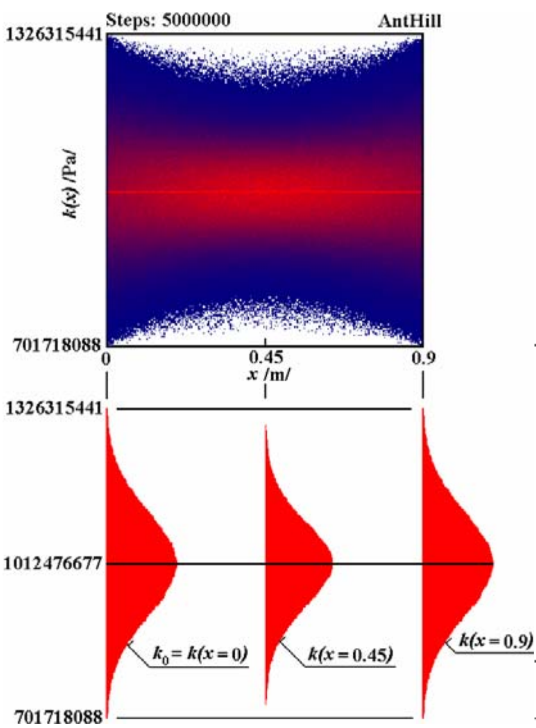


Fig. 11 2D Histogram and its Sections for Output Parameter $k = k(x)$

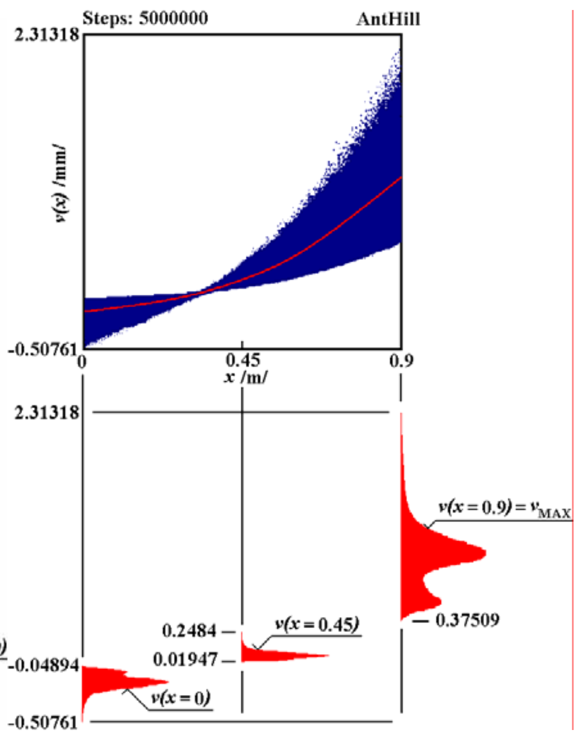


Fig. 12 2D Histogram and its Sections for Output Parameter $v = v(x)$

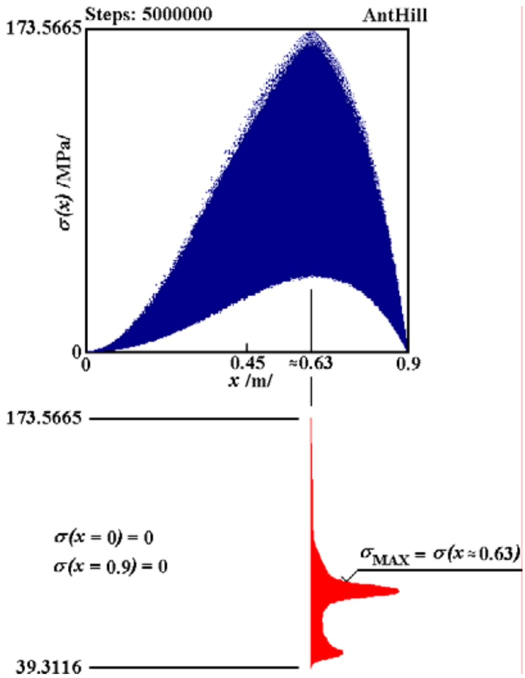


Fig. 13 2D Histogram and its Section for Output Parameter $\sigma = \sigma(x)$

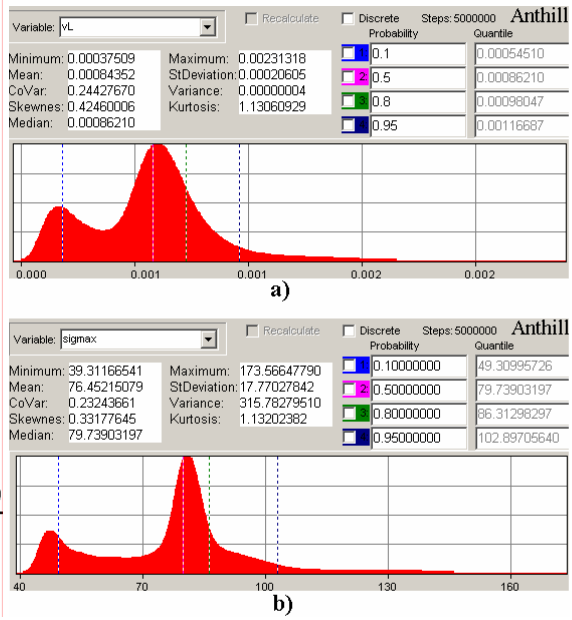


Fig. 14 Histograms of Output Parameters: a) $v_{\mathrm{MAX}} = v(x = \mathrm{L} = 0.9\,\mathrm{m})$, b) $\sigma_{\mathrm{MAX}} = \sigma(x \approx 0.63\,\mathrm{m})$

Tab. 3 Solved Example (Results of AntHill Software)

Output Variables:	Minimum:	Median:	Maximum:	See Figures:
$k(x)$ [Pa]	701718088	1012476677	1326315441	11
v_{MAX} [mm]	3.75×10^{-1}	8.62×10^{-1}	2.31	12 and 14a)
σ_{MAX} [MPa]	39.31	79.74	173.57	13 and 14b)

Probability analysis can be also used for reliability expertise of the beam (AntHill software, SBRA Method). Hence, the function of safety F_S (reliability factor) is defined by:

$$F_S = R_e - \sigma_{\mathrm{MAX}} \,, \tag{8}$$

see also Fig.15 and 16. Hence, it is evident that the safe situation occurs when $F_S > 0$ (i.e. yield stress R_e is greater than maximal bending stress σ_{MAX}).

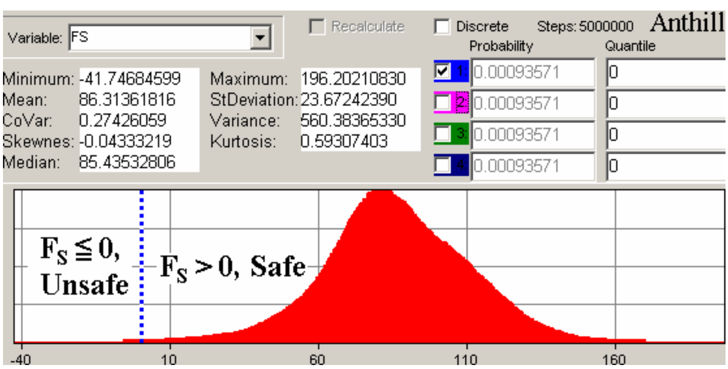


Fig. 15 Histogram of Output Parameters F_S [MPa]

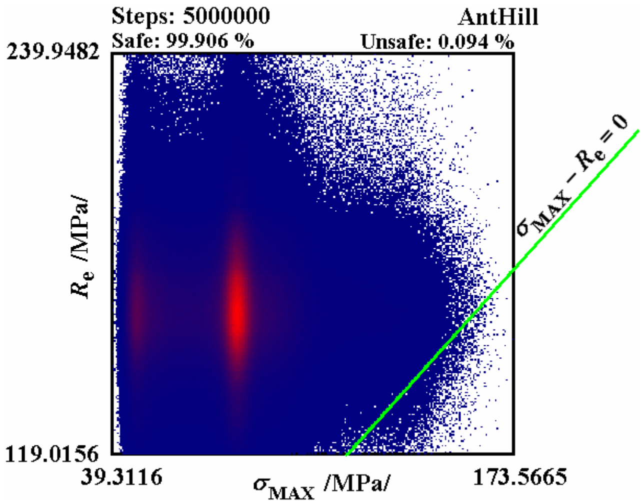


Fig. 16 2D Histogram of Output Parameters For Calculation of F_S

The above function of safety F_S was analyzed by AntHill software. Hence, the probability that $F_S \leq 0$ is 9.3571×10^{-4} (i.e. the yield stress and plastic deformations will be reached with a probability of 9.3571×10^{-4}). In other words, $9.3571 \times 10^{-4} \approx 0.094\,\%$ of all states will result in yielding.

5 CONCLUSIONS

General solution for the chosen beam on elastic foundation with longitudinal changes was derived in the form of polynomial function of 6th order. Derived results were used for probabilistic analysis (SBRA method, Monte Carlo Simulation Method, Anthill software).

Finally, the probability that the plastic deformations occurs in the beam is 0.094%. Figure 16 shows distribution of yield stress versus maximal bending stresses and calculation of safety, which is 99.906%.

Other examples of the applications of SBRA method are shown in references [2], [3], [5] and [6].

Probabilistic expertise of the beam describes better the reality than the clasical deterministic expertise.

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