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LINEARIZATION AND SIMPLIFICATION OF MATHEMATICAL MODEL OF HYDRAULIC SYSTEM WITH VARIABLE VOLUME

LINEARIZACE A ZJEDNODUŠENÍ MATEMATICKÉHO MODELU HYDRAULICKÉHO SYSTÉMU S PROMĚNNÝM OBJEMEM

Abstract

The paper presents the method of linearization of a hydraulic actuator system to a quasi-linear form. The hydraulic actuator system was designed in two forms: as differential equations (accurate model), and as a quasi-linear model (simplified model). The relationship between the parameters of the two models was determined. The simplified model was verified with the accurate model.

Abstrakt

Příspěvek popisuje metodu linearizace modelu hydraulického pohonu na kvazilineární tvar. Model hydraulického pohonu byl navržen ve dvou tvarech: jako diferenciální rovnice (přesný model) a jako kvazilineární model (zjednodušený model). Zjednodušený model byl porovnán s přesným modelem.

1 INTRODUCTION

The aim of this paper is to present the issue of the linearization of a hydraulic actuator model. The linearization of non-linear mechanical systems deals with the replacement of a non-linear motion equation with an approximate linear equation. It is possible only for a certain group of non-linear systems. If, for instance, the system has a characteristic of a continuous function at an operating point and its derivative is different from zero, then, in the operating point neighbourhood, this characteristic can be replaced with a straight line with a slope that is equal to the derivative for the operating point.

Nonlinearities in hydraulic systems result from the physical character of phenomena that occur in them. In most cases nonlinearities depend on quantities of the variables and their derivatives. The mathematical model of the system is described by means of non-linear ordinary differential equations, which makes it difficult to identify or investigate it dynamically with the help of classical methods used in control theory.

Non-linear functions are applied to describe such phenomena as fluid flow through the valve, compressibility, viscosity and thermodynamical effects. Choosing linear mathematical models we consciously accept simplifications in relation to behaviours of the investigated non-linear objects. In automatic control issues, linear dynamic models are the basis for the methods of control laws synthesis.

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2 MATHEMATICAL MODEL

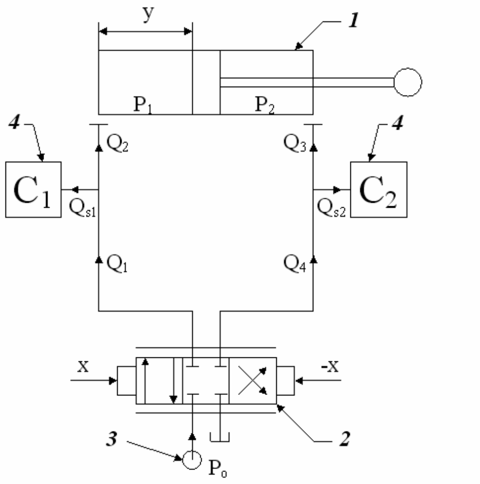

Fig. 1 Hydraulic control system

Illustrated in Figure 1 is a drive system with constant pressure feed p_o, in which a flow operator with a control valve (2) was used as a throttling element, and a hydraulic actuator (1) as a receiving body. Constant pressure p_o was generated by a pump (3). The fluid under the constant pressure enters the manifold (2), which directs it to a particular place in a hydraulic circuit. The system uses accumulators (4), which store the pressurised fluid when the demand for it decreases and give it up when the demand rises, in short time periods. The fluid flows into the hydraulic actuator and, pressing on the area of a cylinder, exerts a force on a piston rod. The fluid in the second chamber of the actuator is pushed and returned to the operator. The shift of the valve's spool is the control signal. It was assumed that the signal may change within the range $-1 \le x \le 1$. For $x = 0$ the orifice is closed. It is fully open when $x = -1$ or $x = 1$. The following simplifying assumptions were adopted when building the hydraulic system: the value of the control signal (input) as a valve' spool shift x is equal to the length of the opened throttling gap, the system is powered with pressure from a constant source, $p_o = \text{const}$, volumetric fluid compressibility modulus is constant within the whole pressure change range in the system, output pressure from the operator into the reservoir is negligibly low in relation to the pressure in the actuator chambers, pressure drops between the pump and the operator equal zero, changes in fluid volume are negligibly small.

With the above-described assumptions the equations of a hydraulic system mathematical model are as follows:

$$v = \dot{y}\,,\,,\quad m\dot{v} + Bv = F_1 p_1 - F_2 p_2$$

$$Q_1 = \alpha x\sqrt{|p_0 - p_1|}H(x) + \alpha x\sqrt{|p_1|}H(-x)\,,\ V_1 = V_{10} + F_1 y\,,\ \frac{V_1}{E_c}\dot{p}_1 = Q_1 - F_1 v - K_v(p_1 - p_2)\,, \tag{1}$$

$$Q_4 = -\alpha x\sqrt{|p_2|}H(x) - \alpha x\sqrt{|p_0 - p_2|}H(-x)\,,\ V_2 = V_{20} - F_2 y\,,\ \frac{V_2}{E_c}\dot{p}_2 = Q_4 + F_2 v + K_v(p_1 - p_2)\,,$$

where:

Q_1, Q_4 – rate of fluid flow through the operator, α – rate coefficient for the fluid flow through the operator's fully open orifice, p_o – feed pressure, $H(x)$ – step function, $H(x) = 1$ for $x \ge 0$,

$H(x)=0$ for $x<0$, V_1 – volume of the part of the system with p_1, V_2 – volume of the part of the system with p_2, V_{10}, V_{20} – initial volumes of the system, E_c – coefficient of flexibility, K_v – volumetric loss coefficient, v – piston speed, m – piston and connected elements mass, B – coefficient of viscous friction, p_1, p_2 – pressure on the actuator's piston, F_1, F_2 – piston areas.

For the simulations presented in the further part of the paper the following nominal values of parameters were adopted

$$\alpha = 1\cdot 10^{-6}\left[m^5/Ns\right],\ p_o = 4.6\cdot 10^7\left[N/m^2\right],\ K_v = 5\cdot 10^{-12}\left[m^5/Ns\right],\ F_1 = 3.5\cdot 10^{-2}\left[m^2\right],$$
$$F_2 = 3\cdot 10^{-2}\left[m^2\right],\ E_c = 1.2\cdot 10^9\left[N/m^2\right],\ V_{10} = 0.01\left[m^3\right],\ V_{20} = 0.02875\left[m^3\right],$$
$$B = 1\cdot 10^5\left[Ns/m\right],\ m = 5\cdot 10^2\left[kg\right]$$

and initial conditions:

$$y(0)=0[m],\ \dot{y}(0)=0[m/s],\ p_1(0)=\frac{F2^3}{F1^3+F2^3}p_o\left[N/m^2\right],\ p_2(0)=\frac{F2^2\cdot F1}{F1^3+F2^3}p_o\left[N/m^2\right].$$

3 LINEARIZATION OF THE HYDRAULIC SYSTEM

Let us note that the equation for the system with an actuator can be expressed as state equations.

$$\dot{z} = Az + b(z)\cdot x\ ,\ v = c^T z\ .$$

Let us assume that the aim is to linearize the equation around an operating point for a defined x and y. First, we determine the operating point from the condition $\dot{z}=0$, that is

$$Az_r + b(z_r)\cdot x = 0$$

When $K_v = 0$, the equation can be solved analytically

$$p_{1n} = \frac{1}{x^2\alpha^2}\left(p_0 x^2\alpha^2 - \frac{F_1^3 p_0 x^2\alpha^2}{F_1^3+F_2^3} - \frac{B^2F_1^2x^4\alpha^4}{2(F_1^3+F_2^3)^2} + \frac{BF_1^2x^2\alpha^2\sqrt{4F_1(F_1^3+F_2^3)p_0x^2\alpha^2+B^2x^4\alpha^4}}{2(F_1^3+F_2^3)^2}\right),$$

$$p_{2n} = \frac{1}{x^2\alpha^2}\left(\frac{F_1F_2^2p_0x^2\alpha^2}{F_1^3+F_2^3} + \frac{B^2F_2^2x^4\alpha^4}{2(F_1^3+F_2^3)^2} - \frac{BF_2^2x^2\alpha^2\sqrt{4F_1(F_1^3+F_2^3)p_0x^2\alpha^2+B^2x^4\alpha^4}}{2(F_1^3+F_2^3)^2}\right),$$

$$v_n = \frac{-Bx^2\alpha^2+\sqrt{4F_1(F_1^3+F_2^3)p_0x^2\alpha^2+B^2x^4\alpha^4}}{2(F_1^3+F_2^3)}.$$

In the general case, when $K_v \neq 0$, the equation can be solved numerically.

At the operating point

$$b(z) = b(z_r) + \frac{\partial b}{\partial z}_{|z=z_r}(z-z_r)$$

Thus

$$\dot{z} = Az + (b(z_r) + \frac{\partial b}{\partial z}(z-z_r))x = \left(A + x\frac{\partial b}{\partial z}\right)z + \left(b(z_r) - \frac{\partial b}{\partial z}z_r\right)x = \widetilde{A}(x,y)z + \widetilde{b}(x,y)x \quad ,$$

$$v = c^T z\,,$$

where

$$c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \beta_1 = \frac{V_1}{E_c},\ \beta_2 = \frac{V_2}{E_c},$$

$$b = \begin{pmatrix} \frac{\alpha}{\beta_1}\sqrt{p_0 - p_1} \\ -\frac{\alpha}{\beta_2}\sqrt{p_2} \\ 0 \end{pmatrix} \text{ for } x \ge 0\,,\ b = \begin{pmatrix} \frac{\alpha}{\beta_1}\sqrt{p_1} \\ -\frac{\alpha}{\beta_2}\sqrt{p_0 - p_2} \\ 0 \end{pmatrix}, \text{ for } x < 0\,.$$

In this way we obtain the transmittance for the defined x and y.

$$\frac{v(s)}{x(s)} = T(s) = c^T\left(sI - \tilde{A}(x,y)\right)^{-1}\tilde{b}(x,y) \tag{2}$$

Let us note that the transmittance coefficients depend on x and y. Transfer function $T(s)$ has possesses one zero and three poles: two complex poles and one real pole.

After the calculation of zeros and poles, it turns out that one of the zeros of a linearized object is close to one of the poles. It is then possible to reduce them and approximate the system by means of lower order system. Indeed, the analysis of the zeros and poles of transfer function, for example, the values of x and y (Table 1) shows that the transfer function zero is close to the real transfer function pole.

Tab. 1 Zero and poles

x	$y\,[m]$	Zero	Real pole	Complex pole	Complex pole
0.1	0.1	-0.553428	-0.602292	-100.373+540.126i	-100.373-540.126i
0.1	0.4	-0.696633	-0.624809	-100.316+491.445i	-100.316-491.445i
1	0.1	-5.5559	-6.04729	-103.749+540.736i	-103.749-540.736i
1	0.4	-6.99936	-6.27291	-103.179+492.015i	-103.179-492.015i

With the defined x and y, dependency $\frac{v(s)}{x(s)}$ can be approximated by means of the second order transmittance. Also, sufficient accuracy is obtained through the approximation of dependencies of transmittance coefficients from x and y in a form of a linear function. Due to the non-symmetry of the actuator ($F_1 \ne F_2$) the coefficients of this function will vary for $x \ge 0$ and $x < 0$.

Thus

$$\begin{aligned} &\ddot{v} + (a^+_{10} + a^+_{11}x + a^+_{12}y + a^+_{13}xy + a^+_{14}y^2)\dot{v} + (a^+_{20} + a^+_{21}x + a^+_{22}y + a^+_{23}y^2)v = \\ &= (b^+_0 + b^+_1 x + b^+_2 y + b^+_3 y^2)x \end{aligned} \qquad \text{for } x \ge 0\,,$$

$$\begin{aligned} &\ddot{v} + (a^-_{10} + a^-_{11}x + a^-_{12}y + a^-_{13}xy + a^-_{14}y^2)\dot{v} + (a^-_{20} + a^-_{21}x + a^-_{22}y + a^-_{23}y^2)v = \\ &= (b^-_0 + b^-_1 x + b^-_2 y + b^-_3 y^2)x \end{aligned} \qquad \text{for } x < 0\,. \tag{3}$$

The values of particular variables are the following:

a_{10}^{+}	a_{11}^{+}	a_{12}^{+}	a_{13}^{+}	a_{14}^{+}
200.762	8.22294	-10.2406	-4.532	20.487
a_{20}^{+}	a_{21}^{+}	a_{22}^{+}	a_{23}^{+}	
363017	1415.21	-657273	932269	
b_{0}^{+}	b_{1}^{+}	b_{2}^{+}	b_{3}^{+}	
63318.5	1.11302	-136751	157578	
a_{10}^{-}	a_{11}^{-}	a_{12}^{-}	a_{13}^{-}	a_{14}^{-}
200.878	-10.0721	-11.5788	8.84922	23.133
a_{20}^{-}	a_{21}^{-}	a_{22}^{-}	a_{23}^{-}	
363146	-1567.8	-657959	932790	
b_{0}^{-}	b_{1}^{-}	b_{2}^{-}	b_{3}^{-}	
38871	-54.1975	-35404.1	106421	

4 SIMULATION

The results of the simulation for real and linearized systems were compared. Figure 2 presents the input signal that is set as a sequence of steps with rising amplitude. It is then possible to compare the systems in a wide range of changes in signal x. Figure. 3 depicts speed chart for both systems. It is clear that the linearized system reproduces the real system in detail.

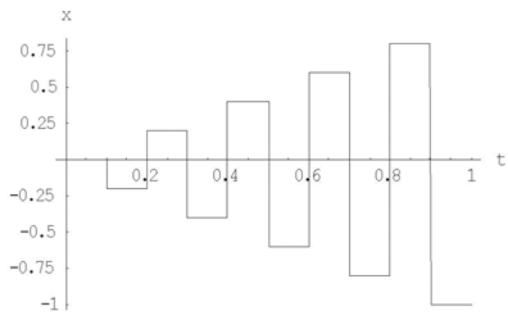

Fig. 2 Input signal x

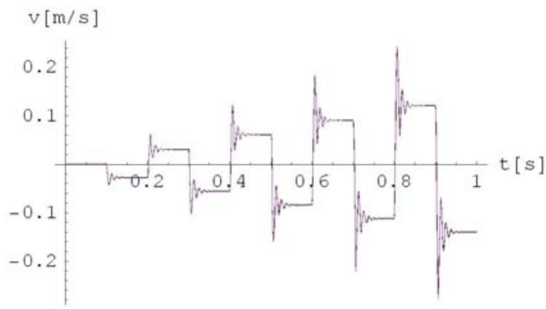

Fig. 3 Piston speed (v) for real and simplified systems

5 APPLICATION

The quasi-linear model (3) with the parameters is obviously another non-linear model. A question arises why one non-linear model is replaced with another. First of all, it should be noted that the quasi-linear model is linear with respect to the system parameters.

This model can be used for identification purposes using the least squares method, which is the most effective method of identification. Another area of application is designing control systems. In order to control a system whose properties depend on the operating points, we can use robust regulators. As their parameters are constant, the system will operate effectively at all operating points. The quasi-linear model can be used to apply a different approach. For a given operating point, x and y, we can determine the regulator using the classic methods of design for linear systems. The regulator parameters depend on the values of x and y.

6 CONCLUSIONS

Having carried out the linearization of an accurate model of a hydraulic actuator, we obtained its simplified form, which reproduces well the behaviour of the real system. For the second order transmittance model (2) both signals coincide. That is why, in this case, the simplification of the mathematical model of an actuator (1) to the quasi-linear form (3) is reasonable. Good results of displacements synchronisation hydraulic systems synthesis are obtained on the basis of simplified linearized mathematical models. It results in their dynamical behaviour being quite similar to the dynamic behaviour of linear systems, thanks to which the synthesis can be conducted based on a linearized model.

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