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A STRONGLY STABLE IMPLEMENTATION OF STATE DERIVATIVE FEEDBACK IN SYSTEM VIBRATION SUPPRESSION

NÁVRH SILNĚ STABILNÍ DERIVAČNÍ STAVOVÉ ZPĚTNÉ VAZBY PRO POTLAČENÍ VIBRACÍ SYSTÉMU

Abstract

The effect of small delays in the state derivative feedback control loop is studied. Such an unusual feedback has proved useful in system vibration suppression, which is used here as a case study example. It is shown that even very small delays at the system inputs can destroy stability of the closed loop system. To prevent such fragility with respect to the delays, the strong stability concept developed for the neutral time delay systems is adopted. It is demonstrated that only such feedback gains can practically be used for which the closed loop system is strongly stable.

Abstrakt

Článek se zabývá studiem vlivu malých dopravních zpoždění na stabilitu systému, který je řízen derivační stavovou zpětnou vazbou. Tuto neobvyklou zpětnou vazbu je možné s výhodou využít k potlačení vibrací systému, jak je ukázáno na prezentovaném aplikačním příkladu. Je také ukázáno, že i velmi malá dopravní zpoždění mohou způsobit nestabilitu uzavřeného regulačního obvodu. K zamezení vzniku takto křehké stability vzhledem ke zpožděním je aplikován koncept silné stability neutrálních systémů. Z praktického hlediska pouze takové koeficienty zpětné vazby mohou být použity pro řízení, pro které je uzavřený regulační obvod silně stabilní.

1 INTRODUCTION

Vibrations may be undesirable in dynamical systems for a host of reasons. They can affect product quality or functionality of products e.g. in the manufacturing of tools. They can affect system performance e.g. in delicate machines that may require vibration isolation. They can affect personal comfort e.g. vibrations in vehicle suspension systems. It is no surprise then that the need commonly arises for such vibrations to be suppressed in dynamical systems.

In vibration control problems, accelerometers are typically used for measuring the system motion. Accelerations are typically the sensed variables as opposed to displacements. When one considers feedback controller implementation, the question of acceleration feedback and indeed state derivative feedback naturally arises. Much attention has been paid to this control problem in the literature. The application of acceleration feedback to vibration suppression problems has been discussed at length in [8,11]. A general pole placement technique for state derivative feedback was proposed in [12] for single-input systems and its application to vibration problems was emphasised. Following from this, the same authors proposed an LQR technique for computing state derivative

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feedback for multiple-input systems in [1]. The application of state derivative feedback to vibration problems has also been discussed from the perspective of robust control in [10].

Even when a dynamical system is modelled as ordinary differential equations, it is extremely important to consider latencies which arise from the application of a control action [6]. Such latencies can occur due to e.g. computational delays, AD-DA conversion, or communication delays. The role of latency phenomena on system stability is of crucial importance when one considers systems with state derivative feedback. It is shown that the application of state derivative feedback renders a system of the neutral type. This induces complications with respect to system stability due to the fact that the system may be very sensitive even to infinitessimal delay changes [14]. For this reason, the notion of strong stability is utilised from [2,3]. This ensures robustness of stability w.r.t. delay perturbations.

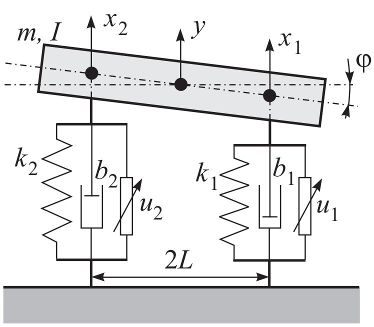
The remainder of the paper is structured as follows: firstly a vibration suppression problem is presented as a motivating example. Then the pertinent issues of state derivative feedback and neutral equations are discussed. Finally, a case study is presented where the results of this study are analysed and compared to those in the literature.

2 CASE STUDY

As the motivation example we adopt the vibration suppression example presented in [1], which is shown in Fig. 1. A linear state space formulation of the system, with states $x = [x_1 \; x_2 \; \dot{x}_1 \; \dot{x}_2]$, assuming φ to be small, is given by

$$\dot{x}(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1c_1 & -k_2c_2 & -b_1c_1 & b_2c_2 \\ -k_1c_2 & -k_2c_1 & -b_1c_2 & b_2c_1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ c_1 & c_2 \\ c_2 & c_1 \end{pmatrix} u(t) \qquad (1)$$

where $c_1 = \frac{1}{m} + \frac{L^2}{I}$, $c_2 = \frac{1}{m} - \frac{L^2}{I}$, $x_3 = 0.5(x_1 + x_2$ and $\varphi = \frac{1}{2L}(x_1 - x_2)$. m and I are the mass and inertia of the mass respectively, k_1 and k_2 are the spring constants, b_1 and b_2 are the damping constants, $2L$ is the distance between the two supports, φ is the angle of inclination of the mass with the horizontal, x_3 is the displacement of the centre of the mass, x_1 and x_2 are the displacements of the sides of the mass and u_1 and u_2 are the control inputs. The objective of applying such control inputs is to interchange or dissipate kinetic and potential energy effectively such that system vibrations are reduced. One way of achieving this is by altering damping and stiffness characteristics of the system e.g. $b_{1,2}$ and $k_{1,2}$. Alternatively the system actuators (piezo-actuators or linear motors) controlled by the inputs *u* can simply displace the system thus counteracting the vibrations present and setting the system to rest.



System parameters:

$m = 10$kg, $I = 1kgm^2$, $L = 1m$,

$k_1 = 500N/m$, $k_2 = 700N/m$,

$b_1 = 10Ns/m$, $b_2 = 20Ns/m$.

Open loop system poles:

$\lambda_{1,2} = 15.1384 \pm 31.1738j$,
$\lambda_{3,4} = 1.3616 \pm 10.7106j$

Fig. 1 Scheme and parameters of the vibration suppression example

The design objective is to achieve vibration suppression using a state derivative feedback controller. In the next section, we provide a short introduction to the theory of state derivative feedback design. Next, we concentrate on the stability consequences of the small uncertain delays appearing in the state derivative feedback loop.

3 INTRODUCTION TO STATE DERIVATIVE FEEDBACK

Let us consider a system of the form

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \tag{2}$$

where $x \in \Re^n$ is the state and $u \in \Re^N$ are the inputs. It is assumed that $\mathbf{A}$ is of full rank, i.e. the system is state controllable and has no root at origin of the complex plane [1]. It is desired to design a stabilising controller of the form

$$u(t) = -\mathbf{K}_{df}\dot{x}(t) \tag{3}$$

The closed loop system then takes the form

$$\dot{x}(t) = (\mathbf{I} + \mathbf{B}\mathbf{K}_{df})^{-1}\mathbf{A}x(t) \tag{4}$$

The design problem is to compute a feedback gain matrix $\mathbf{K}$ such that the closed loop poles are located in the open left half complex plane. This is achieved through a system transformation to the Frobenius canonical form and by applying a technique similar to that applied in the derivation of the Ackermann formula for state feedback. For derivations of the Ackermann formula for state feedback and the equation for state derivative feedback the reader is referred to [9] and [12] respectively. The formula derived in [12] can be generalised, using work presented in [5] to compute state derivative feedback gains for MIMO systems.

It is important to note that a relationship can be derived between state feedback and state derivative feedback. From [1], the following relationship can be derived

$$u(t) = -\mathbf{K}_{sf}x(t) = -\mathbf{K}_{df}\dot{x}(t) = -\mathbf{K}_{df}(\mathbf{A}x(t) + \mathbf{B}u(t)) = -\mathbf{K}_{df}(\mathbf{A} - \mathbf{B}\mathbf{K}_{sf})x(t) \tag{5}$$

So the following relationship can be stated

$$\mathbf{K}_{df} = \mathbf{K}_{sf}(\mathbf{A} - \mathbf{B}\mathbf{K}_{sf})^{-1} \tag{6}$$

This is an interesting result considering the fact that algorithms for calculating state feedback gains are widely available in software packages such as MATLAB [4]. This relationship is used to obtain the results presented in this work, unless otherwise stated. Obviously, under the condition stated in the beginning of this section, the dynamics of the system can be assigned by (3) as if the classical state feedback is used. However, as will be shown in the next section, the state derivative feedback system (4) can be critically fragile w.r.t. small delays in the feedback loop.

4 THE EFFECT OF SMALL DELAYS IN STATE DERIVATIVE FEEDBACK

In real systems, it is inevitable that small uncertain time delays may occur in the feedback loop of an applied control algorithm. Recalling equation (1), and rewriting the equation considering such delays, we obtain the system in the form

$$\dot{x}(t) = \mathbf{A}x(t) + \sum_{j=1}^{N}\mathbf{B}_j u(t - \tau_j) \tag{7}$$

where τ_j are the delays arising in the feedback loop of the system. The summation over the input terms is necessary, due to the fact that each individual input may have a corresponding unique delay value. Now considering the state derivative feedback (3) (for simplicity, we use the notation $\mathbf{K} = \mathbf{K}_{df}$), the closed loop system changes from the form (4) to

$$\dot{x}(t) + \sum_{j=1}^{N} \mathbf{B}_j \mathbf{K}_j \dot{x}(t - \tau_j) = \mathbf{A}x(t) \tag{8}$$

Let us point out that equation of the form (8) is referred to as a neutral differential equation. A characteristic of neutral equations is that delays are present in the derivative of the state. Such a system description induces complications when one considers system stability. The concept of a strongly stable solution, introduced in [3], must therefore be considered.

4.1 Strong stability of neutral systems

For system (8) we can define the associated difference equation given by

$$x(t) + \sum_{j=1}^{N} \mathbf{B}_j \mathbf{K}_j x(t - \tau_j) = 0 \tag{9}$$

Now, let us define the smallest real upper bound of the infinite spectrum of the neutral system (8) as

$$c_N(\vec{\tau}) = \sup \left\{ \Re(\lambda) : \ C(\lambda) = det \left(\lambda \left(\mathbf{I} + \sum_{j=1}^{N} \mathbf{B}_j \mathbf{K}_j e^{-\lambda \tau_j} \right) - A \right) = 0 \right\} \tag{10}$$

where $C(\lambda)$ is the characteristic function of (8). Analogously let us define

$$c_D(\vec{\tau}) = \sup \left\{ \Re(\lambda) : \ D(\lambda) = det \left(\mathbf{I} + \sum_{j=1}^{N} \mathbf{B}_j \mathbf{K}_j e^{-\lambda \tau_j} \right) = 0 \right\} \tag{11}$$

as the smallest real upper bound of the infinite spectrum of the associated difference equation (9), where $D(\lambda)$ is its characteristic function. In both (10) and (11), $\vec{\tau}$ is a vector of the delays present in the system, e.g. $\vec{\tau} = [\tau_1, \tau_2 \ldots \tau_N]$.

It has been shown in [2, 3] see also [7, 14] that

$$c_D(\vec{\tau}) \leq c_N(\vec{\tau}) \tag{12}$$

Therefore, the necessary condition for stability of neutral system (8) is stability of the associated difference equation (9).

It has also been shown in [2,3], see also [7, 14], that although the smallest upper bound (11) is continuous in the coefficients of the matrices $\mathbf{B}_j \mathbf{K}_j$, it is not continuous in the delays $\vec{\tau}$. A major consequence of this non-continuity is that arbitrarily small delay perturbations may destroy stability of the difference equation. Thus, the stability for given fixed delays is not sufficient from the robustness point of view. Therefore the concept of strong stability has been introduced in [2, 3]. It can be stated that the solution of difference equation (10) is strongly stable if it remains stable when subjected to small variations in the delays. In [2,3] the criterion for evaluating the strong stability is derived as follows. The solution of the delay difference equation (10) is strongly stable if and only if

$$\gamma_0 < 1, \gamma_0 := \max_{\vec{\theta} \in [0,\ 2\pi]^m} r_\sigma \left(\sum_{j=1}^{m} \mathbf{B}_j \mathbf{K}_j e^{i\theta_k} \right) \tag{13}$$

As can be seen from (13), the strong stability is independent of the value of the delays. This means that the stability locally in the delays is equivalent with the stability globally in the delays [2, 3, 14].

Let us also remark that if $\gamma_0 > 1$ then equation (9) is unstable for rationally independent[1] delays, [2, 3]. Moreover, in such a case, the closed loop system has infinitely many unstable roots.

5 CASE STUDY RESULTS

In this section, first we study the effects of small delays in the closed loop system (1) - (2) for the feedback gain derived in [1]. Next, we design a feedback gain which provides strongly stable solution of the system.

5.1 Analysing the effect of delays in the vibration suppression example

Recall the system (1) presented in section 2 and consider the stabilising state derivative feedback gain matrix from [1]

$$\mathbf{K}_{df} = \begin{pmatrix} 98.0081 & -6.5270 & 1.5875 & -1.5119 \\ -4.6622 & 35.6412 & -0.2978 & 1.9490 \end{pmatrix} \tag{14}$$

The corresponding roots of the closed loop system are $\lambda_{1,2} = 23.1893 \pm 5.9365j$ and $\lambda_{3,4} = 5.6331 \pm 10.1242j$. The transient response of this delay free closed loop system from an initial state $x_0 = [-0.01, 0.02, -0.02, 0.01]^T$ can be seen in Figure 2-left. The system response shows desirable characteristics, namely a settling time of approximately $0.5s$.

Now the effects of small uncertain delays in the feedback loop are analysed. Consider the vibration system in the form of (8), i.e. with delays at the system inputs. Even though the values of the delays are not known, for demonstration purposes we consider them here as $\vec{\tau} = [0.001\ 0.001\pi]$. Evaluating the strong stability condition according to (13) results in $\gamma_0 = 5.32$. Thus the difference equation associated with the closed loop system is not strongly stable. Notice that the delays are rationally independent, therefore the system is not stable, as can be seen from the responses in Figure 2 - right. This fact can also be verified by computing the spectrum of the system, using e.g. the quasi-polynomial based root-finder (QPmR) presented in [13].

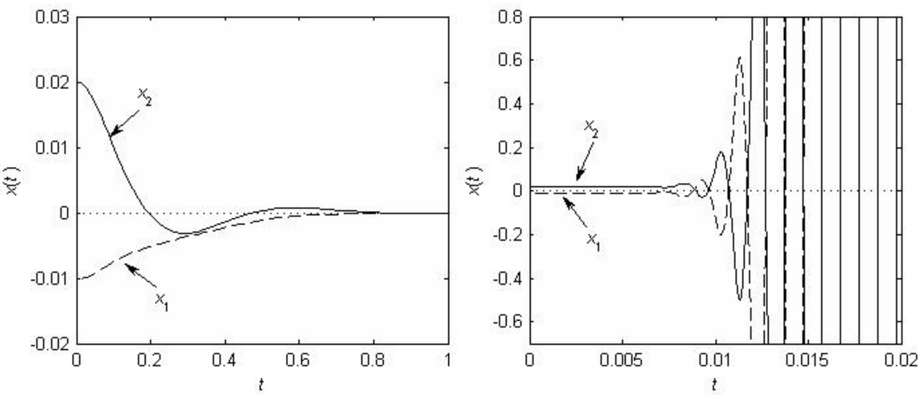


Fig. 2 Response of the closed loop system (1)-(2) with the gain (15): left – no delays, right – delays $\vec{\tau} = [0.001\ 0.001\pi]$ at the inputs.

[1] Delay terms e.g. τ_1 and τ_2 are deemed to rationally independent if their ratio is an irrational number.

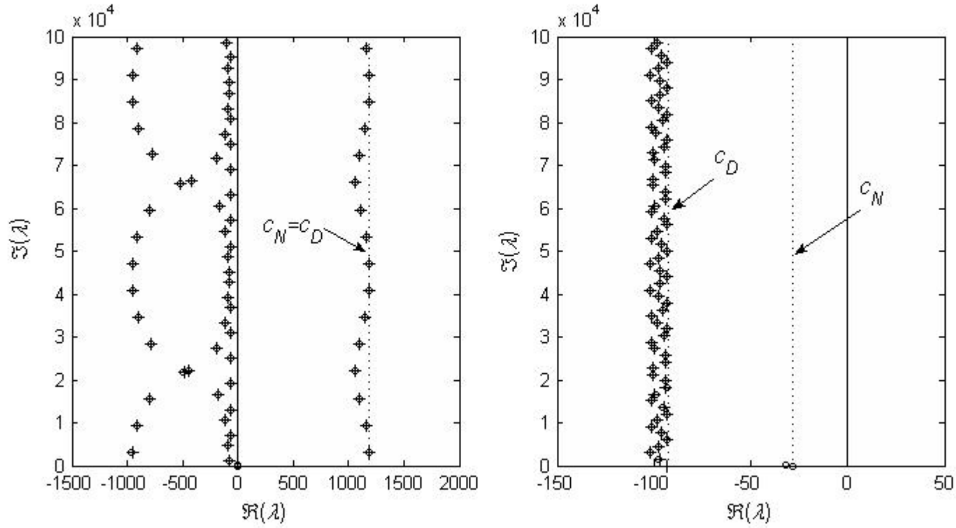


Fig. 3 Spectrum of the closed loop system, considering the effects of delays $\vec{\tau} = [0.001\ 0.001\pi]$ at the inputs, left – using the feedback gain (14), right – using the feedback gain (15) circles - roots of neutral system (8), + - roots of the associated difference equation (9)

The spectrum of the closed loop system can be seen in Figure 3-left. It is evident from the figure that for the given delays, the system is unstable with infinitely many unstable roots. Notice that even though the delays are very small compare to the magnitudes of the roots $\lambda_{1,2}$ and $\lambda_{3,4}$, both smallest upper bounds of the spectra are very large, i.e. $c_N(\vec{\tau}) = c_D(\vec{\tau}) = 1183.4$. Interestingly, as it has been shown in [15] see also [14], the smaller the values of the rationally independent delays, the larger are the upper bounds. It is important to recognise that the algorithms to compute state derivative feedback gains cited in this work [1, 12] do not consider the necessity of a strongly stable solution. However, from the robustness point of view, the feedback gain should satisfy the condition (13), as it is not the case for the gain (14).

5.2 Strongly stable implementation

In this section, a procedure is used to search the root space for potential strongly stable solutions of the state derivative feedback implementation. The result is optimised in the sense that the solution with greatest robustness to small delay perturbations, i.e. the solution with $\min(\gamma_0)$, is chosen. The procedure which is used can be summarised as follows: First, bounds on $\Re(\lambda)$ are selected as an upper bound of $\Re(\lambda) = -5$ and lower bound of $\Re(\lambda) = -35$. Then the damping constraints corresponding to the line $|\Im(\lambda)| = -\Re(\lambda)$ are imposed on the search space. Over the region defined by these bounds, the mesh-grid is defined with the grid-step of $d = 1$. Furthermore for all possible combinations of the root positions over the grid points defined above, the feedback gain K_{df} is computed using the pole placement procedure. In the interest of computational simplicity, the roots of the solution are chosen to be complex conjugates only. The MATLAB algorithm *place.m* [4] is used to compute the state feedback gain at each combination of the pole positions and then the corresponding state derivative feedback gains are computed using the relationship in equation (6). Besides, also the strong stability quantity γ_0 is computed for each $\mathbf{K}_{df}$.

It is interesting to consider the variation of γ_0 w.r.t. changes in root locations. In Figure 4, γ_0 is plotted as a function of $\Re(\lambda_{3,4})$ and $|\Im(\lambda_{3,4})|$ while $\lambda_{1,2}$ are held constant at $-23 \pm 17i$. It is interesting to note that only an area of high frequency poles close to the origin give rise to an exponentially unstable solution. This information would suggest that the solution can be arbitrarily

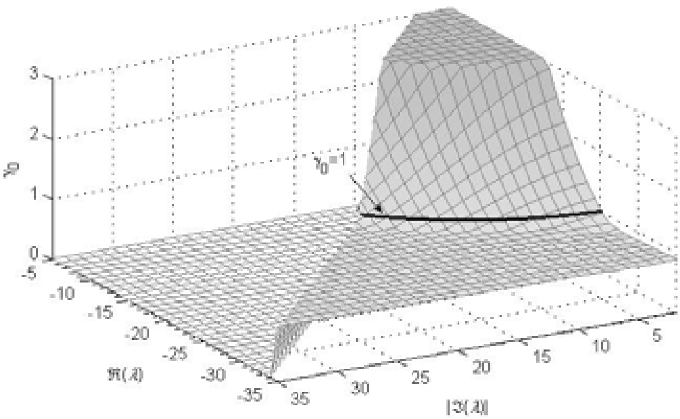


Fig. 4 Variation of γ_0 w.r.t. one pair $\lambda_{3,4}$ of complex conjugate system roots over the defined mesh grid, while $\lambda_{1,2}$=$-23 \pm 17i$. Notice that due to proper scaling of the figure, γ_0 surface is cut by the level $\gamma_0 = 3$. However, obviously γ_0 attains much larger values at the cut region

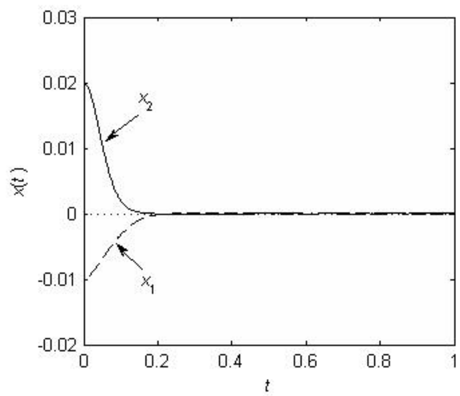


Fig. 5 Response of the closed loop system (1)-(2) with the gain (15) and the delays $\vec{\tau} = [0.001\ 0.001\pi]$ at the inputs

chosen outside of this region. In the interest of solution robustness however, the solution with $min(\gamma_0) = 0.8191$ is chosen instead, for which $\lambda_{1,2} = -20 \pm 11j$, $\lambda_{3,4} = -23 \pm 17j$ and

$$\mathbf{K}_{df} = \begin{pmatrix} 28.3877 & -0.0038 & -1.7903 & -2.2501 \\ -0.0053 & 19.3643 & -2.2502 & -1.8943 \end{pmatrix} \tag{15}$$

As can be seen in Figure 5, the closed loop system shows good response characteristics, even in the presence of small time delays. A settling time of approximately $0.2s$ is observed. Let us remark that the delay free closed loop system has almost identical responses as those shown in Figure 5 and therefore they are not presented here. As a basis for comparison with the results obtained by (14), the spectrum of the closed loop system under the influence of the small time delays is presented in Figure 3-right. Obviously the infinitely many roots are bounded from the right by the upper bound $c_D(\vec{\tau}) = -92.26$, which is safely far from the stability boundary.

6 CONCLUSIONS

In the paper, we demonstrated on the vibration suppression example that the stability achieved by the state derivative feedback can be dangerously fragile with respect to small delays in the feedback loop. It has been shown that even negligibly small delays can introduce infinitely many roots to the right half of the complex plane. Therefore, to obtain a practically applicable solution, the resulting dynamics need to be checked using the theory of strong stability developed for neutral systems. Only such feedback gains are acceptable for the state derivative feedback, for which the closed loop system is strongly stable.

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