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USING KALMAN FILTER FOR DIGITAL SIGNAL PROCESSORS

POUŽITÍ KALMANOVA FILTRU PRO DIGITÁLNÍ SIGNÁLOVÉ PROCESORY

Abstract

This distribution deals with the description of the created application which is dedicated for the calculation of constant value estimation of a measured signal. This signal is loaded by the measurement noise. The using of an Adaptive Kalman filter is the core of the created application. The value of selective variance of a measurement signal is used as an adaptable parameter of Kalman filter. This application was implemented into Digital Signal Processor (DSP) Environment, in the concrete 32bit ADSP-21065L DSP processor was used. This processor enables to process data in the floating-point format.

Abstrakt

Tento příspěvek se zabývá popisem vytvořené aplikační úlohy, která je určena pro výpočet odhadu konstantní hodnoty z měřeného signálu, který je zatížen šumem měření. Jádrem vytvořené aplikační úlohy je použití adaptivního Kalmanova filtru. Velikost výběrového rozptylu šumu měření je použita jako adaptabilní parametr Kalmanova filtru. Tato aplikační úloha byla implementována v prostředí digitálních signálových procesorů (DSP), konkrétně byl použit 32-bitový ADSP-21065L DSP procesor, který umožňuje zpracovávat data ve formátu s plovoucí řádovou čárkou.

1 INTRODUCTION

In 1960, R. E. Kalman published his famous work [Kalman, 1960] describing the recursive solution problem of the linear filtration of discrete data. Kalman filter is the set of mathematic equations which offer very efficient recursive solution given by least-square method. This filter is very efficient in many aspects. It can estimate past, present and even future states.

2 KALMAN EQUATIONS

Kalman equations solve the problem of an optimal state estimation, $\mathbf{x} \in \Re^n$, of a linear discrete stochastic system which can be described by steady-state equations (1) and (2).

$$\mathbf{x}_k = \mathbf{A}_{k,k-1} \cdot \mathbf{x}_{k-1} + \mathbf{v}_{k-1}, \tag{1}$$

$$\mathbf{y}_k = \mathbf{C}_k \cdot \mathbf{x}_k + \mathbf{w}_k, \tag{2}$$

where

$\mathbf{v}_k, \mathbf{w}_k$ – are sampled stochastic sequence with normal distribution and with the character of white noise.

Their names are the process noise and the measurement noise. Kalman equations for mentioned linear discrete stochactic system (1) and (2) are followig.

$$\hat{\mathbf{x}}_k^{'} = \mathbf{A}_{k,k-1} \cdot \hat{\mathbf{x}}_{k-1}, \tag{3}$$

$$\mathbf{P}_k^{'} = \mathbf{A}_{k,k-1} \cdot \mathbf{P}_{k-1} \cdot \mathbf{A}_{k,k-1}^T + \mathbf{Q}_{k-1}, \tag{4}$$

$$\mathbf{K}_k = \mathbf{P}_k^{'} \cdot \mathbf{C}_k^T \cdot \left(\mathbf{C}_k \cdot \mathbf{P}_k^{'} \cdot \mathbf{C}_k^T + \mathbf{R}_k\right)^{-1}, \tag{5}$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^{'} + \mathbf{K}_k \cdot \left(\mathbf{y}_k - \mathbf{C}_k \cdot \hat{\mathbf{x}}_k^{'}\right), \tag{6}$$

$$\mathbf{P}_k = \mathbf{P}_k^{'} - \mathbf{K}_k \cdot \mathbf{C}_k \cdot \mathbf{P}_k^{'}, \tag{7}$$

where

$\hat{\mathbf{x}}_k^{'}$ – is an apriori state estimation,

$\hat{\mathbf{x}}_k$ – is an aposteriori state estimation,

$\mathbf{P}_k^{'}$ – is a covariance matrix of apriori state estimation errors,

$\mathbf{P}_k$ – is a covariance matrix of aposteriori state estimation errors,

$\mathbf{K}_k$ – is Kalman gain,

$\mathbf{Q}$ – is a covariance matrix of the process noise,

$\mathbf{R}$ – is a covariance matrix of the measurement noise.

Kalman equations present an algorithm, which generates a sequence of linear state estimations (apriori and aposteriori) and a sequence of covariance matrices of state estimation errors. This algorithm is created by two subsets. The first subset is produced by equations (3) and (4). These relations determine an apriori state estimation $\hat{\mathbf{x}}_k^{'}$ in the *k*th measurement step and an appropriate covariance error matrix $\mathbf{P}_k^{'}$ of this an apriori state estimation. These two equation are called the time or prediction algorithm step. The second subset is produced by equations (5) – (7). These relations determine an aposteriori state estimation $\hat{x}_k$, Kalman gain $\mathbf{K}_k$ in the *k*th step and an appropriate covariance error matrix $\mathbf{P}_k$ of this an aposteriori state estimation. These three equations are called the date or correction algorithm step. This algorithm is shown in Fig. 1.

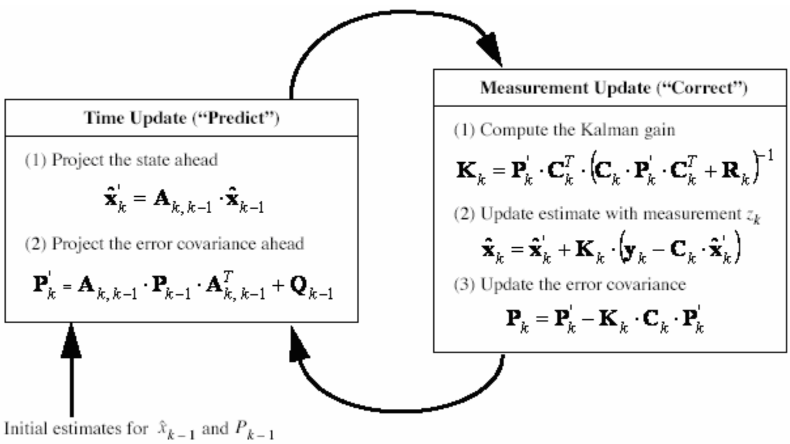

Fig. 1 Algorithm of Kalman filter

Meaning of Kalman gain

In professional literature, Kalman gain $\mathbf{K}_k$ (5) is derived as the minimization of a covariance error matrix of an aposteriori state estimation $\mathbf{P}_k$. The influence of Kalman gain is following. If a covariance matrix of the measurement noise $\mathbf{R}$ approaches zero, then Kalman gain $\mathbf{K}_k$ has greater influence on a value of a covariance error matrix of an aposteriori state estimation $\mathbf{P}_k$.

$$\lim_{\mathbf{R}_k \to \mathbf{0}} \mathbf{K}_k = \mathbf{C}_k^{-1} \qquad (8)$$

On the other side, if a covariance error matrix of an apriori state estimation $\mathbf{P}_k^{'}$ approaches zero, then Kalman gain $\mathbf{K}_k$ has smaller influence.

$$\lim_{\mathbf{P}_k^{'} \to \mathbf{0}} \mathbf{K}_k = \mathbf{0} \qquad (9)$$

If a covariance matrix of the measurement noise $\mathbf{R}$ approaches zero, then actual measurement $\mathbf{y}_k$ is trusted more and more, while predicted measurement $\mathbf{C}_k \cdot \hat{\mathbf{x}}_k^{'}$ is trusted less and less. On the other side, if a covariance error matrix of an apriori state estimation $\mathbf{P}_k^{'}$ approaches zero, then actual measurement $\mathbf{y}_k$ is trusted less and less, while predicted measurement $\mathbf{C}_k \cdot \hat{\mathbf{x}}_k^{'}$ is trusted more and more.

Meaning of covariance matrices of the process and measurement noise

During the creation of an application that uses Kalman equations, it is necessary to define a concrete form of covariance matrix of the process noise $\mathbf{Q}$ and of covariance matrix of the measurement noise $\mathbf{R}$. The random variables $\mathbf{w}_k$ and $\mathbf{v}_k$ represent the process and measurement noise. They are assumed to be independent (of each other), white, and with normal probability distributions. These covariance matrices are diagonal, their elements on the main diagonal are equal to variance of these signals.

While in practise it isn't problem to get measurement noise $\mathbf{w}_k$ and evaluate the estimation of variance value and to establish the covariance matrix of the measurement noise $\mathbf{R}$, it can be much more difficult to establish the covariance matrix of the process noise $\mathbf{Q}$. Mostly by its usage we can describe the uncertainty between real process and a process described by steady-state model. In practise the establishing of concrete forms of these matrices is taken place on measured data. Of course establishing of these matrices can take place during the filtration process.

3 RECURSION FORMULA

The adaptability of the created application reposes on transient computation of an estimation of variance value of measurement noise $\mathbf{w}_k$ during the filtration process. The covariance matrix of the measurement noise $\mathbf{R}$ is continuously updated. So this application can be use also for processes with non-stationary measurement noise $\mathbf{w}_k$.

Numerical characteristics of random values are computed on the basis of knowledge of distribution type and values of their parameters. It is assumed that the measurement noise $\mathbf{w}_k$ is normal probability distributions. This distribution has two parameters, i.e. mean value μ and variance σ^2.

These parameters are very often derived from measured data. But we treat with finite set of random realization that is called random selection. This random selection represents a sequence of independent and equally distributed random values $X_1, X_2, \ldots, X_N$, where N is selection length.

From a random selection we can compute the selection average $\overline{X}$ (10) and the selection variance m_2 (11), whereas $\sqrt{m_2}$ is the selection standard deviation. For better limit properties we can use S^2 (12).

$$\overline{X} = \frac{1}{N} \cdot \sum_{i=1}^{N} X_i \tag{10}$$

$$m_2 = \frac{1}{N} \cdot \sum_{i=1}^{N} \left(X_i - \overline{X}\right)^2 \tag{11}$$

$$S^2 = \frac{1}{N-1} \cdot \sum_{i=1}^{N} \left(X_i - \overline{X}\right)^2 \tag{12}$$

The selection average and the selection variance are random values for which we can compute also their numerical characteristics, for example the mean value and variance. Let the distribution of values from a selection has mean value μ and variance σ^2. It is desirable that the mean value of a selection average would be equal to mean value of the selection values, i.e. $E\left\{\overline{X}\right\} = \mu$. This property of characteristic is called the unbiased estimation of an appropriate parameter. If the mean value of a selection average isn't equal to mean value of random value, then this estimation is biased. It can be proved that S^2 is the unbiased variance estimation, i.e. $E\left\{S^2\right\} = \sigma^2$. The further property is that estimation variance decreases with increasing length of selection. It can be proved that $D\left\{\overline{X}\right\} = \sigma^2 / N$.

Applications dedicated for using in digital signal processors environment cannot by based on formulas like (10)-(12). Digital signal processing in these systems cannot be provided with measured data. It would require the storing relatively very huge content of measured data into memory and also time delay that is mainly given by the acquiring time. General disadvantage of these formulas is necessity to have to disposition the given record of measured signal. So it was necessary to find an approach how to continuously compute the selection average and the selection variance.

Let $\overline{X}_{N-1}$ is the selection average for a record which contains $N-1$ samples and let $\overline{X}_N$ is the selection average for a record which contains N samples.

$$\overline{X}_{N-1} = \frac{1}{N-1} \cdot \sum_{i=1}^{N-1} X_i \tag{13}$$

$$\overline{X}_N = \frac{1}{N} \cdot \sum_{i=1}^{N} X_i \tag{14}$$

Let the difference between the selection average $\overline{X}_{N-1}$ and the selection average $\overline{X}_N$ is denoted Δ_1.

$$\overline{X}_N = \overline{X}_{N-1} + \Delta_1 \tag{15}$$

The formula for determination of this difference can be derived by the following consecution.

$$\Delta_1 = \overline{X}_N - \overline{X}_{N-1} = \frac{1}{N} \cdot \sum_{i=N-1}^{N} X_i - \frac{1}{N-1} \cdot \sum_{i=1}^{N-1} X_i = \frac{1}{N} \cdot \left(\sum_{i=1}^{N-1} X_i + X_N \right) - \frac{1}{N-1} \cdot \sum_{i=1}^{N-1} X_i =$$

$$= \frac{1}{N \cdot (N-1)} \cdot \left[(N-1) \cdot \sum_{i=1}^{N-1} X_i + (N-1) \cdot X_N - N \cdot \sum_{i=1}^{N-1} X_i \right] =$$

$$= \frac{1}{N} \cdot X_N - \frac{1}{N \cdot (N-1)} \cdot \sum_{i=1}^{N-1} X_i = \frac{1}{N} \cdot \left(X_N - \overline{X}_{N-1} \right)$$

$$\Delta_1 = \frac{1}{N} \cdot \left(X_N - \overline{X}_{N-1} \right) \tag{16}$$

Substituting (16) into (15) we get the formula for computing the selection average for a record that contains N samples (17). It can be done because of knowing of previous value of the selection average that contains $N-1$ samples and of computing of an incremental difference. The advantage of this approach is that we don't have to store measured data.

$$\overline{X}_N = \overline{X}_{N-1} + \frac{1}{N} \cdot \left(X_N - \overline{X}_{N-1} \right) \tag{17}$$

By the same approach we can find a formula for determination of the selection variance $m_{2,N}$ for a record that contains N samples (18) on the basis of knowing the previous value of the selection variance $m_{2,N-1}$ that contains $N-1$ samples and of computing of an incremental difference. The advantage of this approach is also that we don't have to store measured data. The derivation of this formula comes out from the following approach. Let the incremental difference is denoted Δ_2.

$$m_{2,N} = m_{2,N-1} + \Delta_2 \tag{18}$$

The formula for determination of this difference can be done by the following approach.

$$\Delta_2 = \frac{1}{N}\sum_{i=1}^{N}\left(X_i - \overline{X}_N\right)^2 - \frac{1}{N-1}\sum_{i=1}^{N-1}\left(X_i - \overline{X}_{N-1}\right)^2 =$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left(X_i^2 - 2\,X_i\,\overline{X}_N + \overline{X}_N^2\right) - \frac{1}{N-1}\sum_{i=1}^{N-1}\left(X_i^2 - 2\,X_i\,\overline{X}_{N-1} + \overline{X}_{N-1}^2\right) =$$

$$= \frac{1}{N}\sum_{i=1}^{N}X_i^2 - \frac{2\,\overline{X}_N}{N}\sum_{i=1}^{N}X_i + \frac{\overline{X}_N^2}{N}N - \frac{1}{N-1}\sum_{i=1}^{N-1}X_i^2 + \frac{2\,\overline{X}_{N-1}}{N-1}\sum_{i=1}^{N-1}X_i - \frac{\overline{X}_{N-1}^2}{N-1}\left(N-1\right) =$$

$$= \frac{1}{N}\left(\sum_{i=1}^{N-1}X_i^2 + X_N^2\right) - 2\,\overline{X}_N^2 + \overline{X}_N^2 - \frac{1}{N-1}\sum_{i=1}^{N-1}X_i^2 + 2\,\overline{X}_{N-1}^2 - \overline{X}_{N-1}^2 =$$

$$= \frac{1}{N}\sum_{i=1}^{N-1}X_i^2 + \frac{X_N^2}{N} - \overline{X}_N^2 - \frac{1}{N-1}\sum_{i=1}^{N-1}X_i^2 + \overline{X}_{N-1}^2 =$$

$$= \frac{\left(N-1\right)\sum_{i=1}^{N-1}X_i^2 - N\sum_{i=1}^{N-1}X_i^2}{N\left(N-1\right)} + \frac{X_N^2}{N} - \left[\overline{X}_{N-1} + \frac{1}{N}\left(X_N - \overline{X}_{N-1}\right)\right]^2 + \overline{X}_{N-1}^2 =$$

$$= \frac{1}{N}\left[X_N^2 - \frac{1}{N-1}\sum_{i=1}^{N-1}X_i^2 - 2\,\overline{X}_{N-1}\left(X_N - \overline{X}_{N-1}\right) - \frac{1}{N}\left(X_N - \overline{X}_{N-1}\right)^2\right]$$

$$\Delta_2 = \frac{1}{N}\left[X_N^2 - \frac{1}{N-1}\sum_{i=1}^{N-1}X_i^2 - 2\,\overline{X}_{N-1}\left(X_N - \overline{X}_{N-1}\right) - \frac{1}{N}\left(X_N - \overline{X}_{N-1}\right)^2\right] \tag{19}$$

4 DESCRIPTION OF THE CREATED APPLICATION

The created application is dedicated for an estimation of random constant that can be for example represented as voltage level. This application assumes the usage of linear discrete stochastic system described by steady-state model (20) and (21).

$$x_k = x_{k-1} + v_{k-1} \tag{20}$$

$$y_k = x_k + w_k \tag{21}$$

So we deal with one-dimensional system, i.e. matrices **A** and **C** transfers into scalars. In addition let $A = C = 1$. Kalman equations will have the following forms

1. **Discrete Kalman filter time update equations**

$$\hat{x}_k^{'} = \hat{x}_{k-1} \tag{22}$$

$$P_k^{'} = P_{k-1} + Q_{k-1} \tag{23}$$

2. **Discrete Kalman filter measurement update equations**

$$K_k = P_k^{'} \cdot \left(P_k^{'} + R_k\right)^{-1} \tag{24}$$

$$\hat{x}_k = \hat{x}_k^{'} + K_k \cdot \left(y_k - \hat{x}_k^{'}\right) \tag{25}$$

$$P_k = P_k^{'} - K_k \cdot P_k^{'} \qquad (26)$$

Suppose very little deviation of the process noise v, let the covariance matrix (respectively the scalar) of the process noise $\mathbf{Q}$ is $Q = 10^{-5}$ V (respectively $Q = 10^{-1}$ V for the third presented example). So we can introduce a certain uncertainty in description of a real system by using the steady-state model (20) and (21).

The created application demonstrates the using of an adaptive Kalman filter. In this case we will change the measurement noise variance w that affects the measuring of required signal. The measurement noise w is generated as a signal with normal probability distribution and it can be presented as a covariance matrix (respectively the scalar) of the measurement noise $\mathbf{R}$.

It is also necessary to establish or estimate the value of the initial aposteriori state estimation $\hat{x}_0$ and the initial covariance matrix (respectively the scalar) of this aposteriori estimation P_0. Let $\hat{x}_0 = 0$ and $P_0 = 1$.

Let the true value of measured signal is constant and 10 V. This signal is corrupted by the measurement noise w with variance $\sigma_w^2 \cong 0.1, 1, 10$ V, respectively. Let the initial value equals zero, i.e. $\sigma_w^2 = R = 0$ V.

The created application has to estimate the constant voltage level, the formulas (15)-(19) are used for the computation of the selection average value $\overline{X}$ and the selection variance m_2. These parameters are used as the adaptable parameter of Kalman filter (22)-(26). The calculated selection variance m_2 is used as parameter R in form (24) for Kalman gain computation, i.e. $m_2 \rightarrow R$. The record length for computation of $\overline{X}$ and m_2 is $N = 256$. It isn't necessary to store these 256 values into a data memory of a digital signal processor.

5 EXPERIMENTAL VERIFICATION

In Fig. 2-4 the results of using the adaptable Kalman filter are presented. It is used random constant estimation from the measured signal that is corrupted by the measurement noise. In Fig. 2 is shown the result of filtration for $\sigma_w^2 \cong 0{,}1$ V, in fig. 3 for $\sigma_w^2 \cong 1$ V and in fig. 4 for $\sigma_w^2 \cong 10$ V.

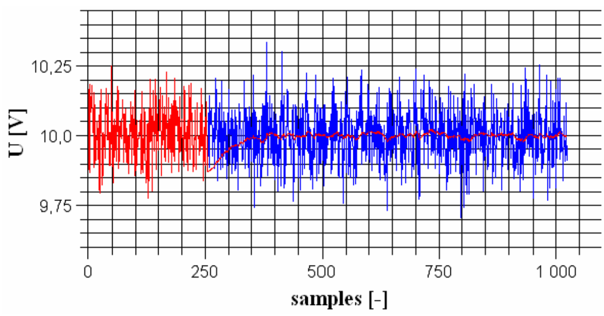

Fig. 2 Using an adaptable Kalman filter, $\sigma_w^2 \cong 0{,}1$ V

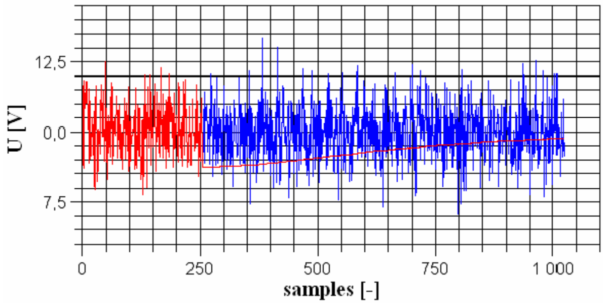

Fig. 3 Using an adaptable Kalman filter, $\sigma_w^2 \cong 1$ V

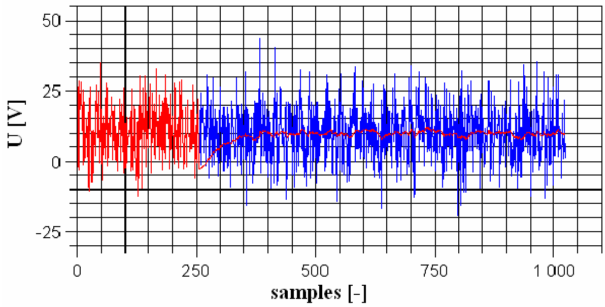

Fig. 3 Using an adaptable Kalman filter, $\sigma_w^2 \cong 10$ V

6 CONCLUSIONS

The created software was implemented into the digital signal processor ADSP-21065L. It can provide estimation of a random constant. It is based on using of Kalman equations. One parameter of these equations is continuously updated. This parameter is a variable that represents the estimation of the measurement noise variance. The variance of this signal is computed for the derivate forms for the selection average computation and for the selection variance computation. The advantage of this approach is that it isn't necessary to store a measured signal into a data memory.

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