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NONLINEAR SYSTEMS: GLOBAL THEORY OF ATTRACTORS AND THEIR TOPOLOGY

NELINEÁRNÍ SYSTÉMY: VŠEOBECNÁ TEORIE ATRAKTORŮ A JEJICH TOPOLOGIE

Abstract

In this paper we shall study the global theory of invariant sets of nonlinear dynamical systems by using a triangulation of the manifold on which the system is defined and replacing the system by a linear model defined by a binary matrix.

Abstrakt

V příspěvku je popsána všeobecná teorie invariatních sad nelineárních dynamických systémů s využitím triangulace agregátu, na kterém je systém definován a výměnou systému pomocí lineárního modelu definovaného binární maticí.

1 INTRODUCTION

In this paper we consider a dynamical system on a closed (compact) triangularsable manifold M and show how to reduce the study of the topology of its invariant sets to a certain linear dynamical system defined by a matrix. From this matrix, it will be possible to obtain approximations to the invariant sets of the original dynamics and then to compute topological invariants of these approximations. We shall be interested in studying the limits of the approximations as the size (mesh) of the triangulation tends to zero.

There are various other approaches to piecewise-smooth approximations-see, for example, [Hsu, 1985] where cellular decompositions are used or [Banks and Khathur, 1989] where piecewise-linear systems are studied via algebraic topology.

The general structure theory of dynamical systems on compact manifolds can be found in [Shilnikov, et al, 1998] or [Banks and Song, 2006] where automorphic function theory is used to generate systems on surfaces. The interpretation given here in terms of a matrix and the related linear discrete system has several advantages. In particular, it is easy to identify invariant sets.

2 BASIC APPROXIMATION & MATRIX REPRESENTATION

Suppose that K is a simplicial complex defining a triangulation of an n-dimensional compact manifold M. Let K_n denote the n-skeleton of K, so that

$$K_n = \{T_1, \cdots, T_L\}$$

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where each T_i is a (closed) n-simplex. We shall define a discrete dynamical system on K_n which approximates the original one. To do this, let $\{b_1,\cdots,b_L\}$ be the barycentres of the simplices of K_n, and let

$$\varphi_i b_i$$

be the solution of the dynamical system through b_i. Suppose that when $\varphi_i b_i$ leaves T_i, it enters T_j, then we define the map

$$FT_i = T_j \tag{1}$$

(Of course, if $\varphi_i b_i$ never leaves T_i, then $FT_i = T_i$.)

Thus we obtain a map

$$F : K_n \to K_n \tag{2}$$

defined by (1). It is a finite, discrete, dynamical system. As such, we can make all the usual definitions (of limit cycles, invariant sets, etc.).

Thus, for example,

Definition A (forward) *invariant set* of the dynamical map F is a subset $S \subseteq K_n$ such that

$$F(S) \subseteq S.$$

We are interested in the question of when invariant sets of the original system are approximated by those of the discrete system (2). In order to answer this, we need to consider the notion of inverse system.

Definition An *inverse system* is a collection

$$X = (X_\lambda, p_{\lambda\lambda}, \Lambda)$$

where $(\Lambda, \leq)$ is a directed set, $X_\lambda, \lambda \in \Lambda$ are some sets and

$$p_{\lambda\mu} : X_\mu \to X_\lambda$$

is a map for $\lambda \leq \mu$, subject to the conditions

$$p_{\lambda\lambda} = id, \ \ \lambda \in \Lambda$$

$$p_{\lambda\mu} p_{\mu\nu} = p_{\lambda\nu}, \ \ \lambda \leq \mu \leq \nu.$$

The *limit X* of the inverse system X is the subset of ΠX_λ consisting of all $x = (x_\lambda) \in \Pi X_\lambda$ such that

$$x_\lambda = p_{\lambda\mu}(x_\mu), \text{ for } \lambda \leq \mu.$$

Note that there are natural projections:

$$p_\lambda : X \to X_\lambda.$$

Given an attractor $\Im$ in a dynamical system, let $U_\Im$ be a neighbourhood of $\Im$ in which all trajectories are attracted to $\Im$. Consider the set of all open coverings C of $\Im$ which are contained in $U_\Im$. Let

$$X = (X_\lambda, p_{\lambda\lambda}, \Lambda)$$

be a directed system where $X_\lambda \in C$, $\lambda \in \Lambda$, X_μ is a refinement of X_λ if $\lambda \le \mu$ and

$$p_{\lambda\mu} : X_\mu \to X_\lambda$$

takes any set in X_μ into one containing it. (We shall insist that the diameters of the sets on a cover decrease with increasing λ.) We can think of the inverse limit as the intersection of all $\cup X_\lambda$, i.e.

$$X = \bigcap_\lambda (\cup X_\lambda).$$

It is in this sense that we say that a covering of a set by some triangulate neighbourhood converges to the set. Note that the inverse limit of such a system of open covers of a metric compactum converges to the set itself. In fact, a more detailed discussion of this theory could be given in terms of strong shape theory as given in the latter reference.

Hence we have

Theorem 1 As the mesh δ of the triangulation K_n tends to zero, a forward invariant set of F tends to the inverse limit X of the above neighbourhood $U_\Im$ of the dynamical attractor $\Im$.

We will introduce a matrix representation of the discrete map (1). In fact, we define a matrix $A \in R^{n\times n}$ by

$$a_{i,j} = \begin{cases} 1 & if\ FT_j = T_i \\ 0 & otherwise \end{cases}$$

We will also identify T_i with the i^{th} unit basis vector of R^n, i.e.

$$T_i \leftrightarrow (0,0,\cdots,0,1,0,\cdots,0)^T.$$

Note that the matrix A has a single '1' in each column, and the other elements of the column are zero. It is easy to see that all powers of A, A^k, $k \ge 1$ also have the same property. Clearly, in terms of A, we have

$$i \to \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = A \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j$$

if $FT_j = T_i$, so that the system is effectively linearised by A.

Theorem 2 If $\{T_{i1}, \cdots, T_{it}\}$ is an invariant set of F, then A is equivalent (by similarity) to a matrix of the form

$$\widetilde{A} = \begin{pmatrix} \widetilde{A}_1 & \widetilde{A}_2 \\ 0 & \widetilde{A}_3 \end{pmatrix}.$$

Proof By renumbering the triangles, if necessary, we can arrange that $\{T_1,\cdots,T_l\}$ is an invariant set. Such a renumbering is equivalent to a similarity transformation on A (composed of elementary operations). The form of A given in the statement then follows from standard representation theory.

Corollary If the dynamical system has a finite number of stable attractors, then, for large enough k, A^k has the form

$$A^k = \begin{pmatrix} \tilde{A}_1 & & & \\ & \ddots & & B_1 \\ & & \tilde{A}_s & \\ & 0 & & B_2 \end{pmatrix}$$

where the off-diagonal blocks are zero.

3 STABILITY THEORY & CONTROL THEORY

We can also use the matrix representation of a system defined by a triangulation of the space to study stability. If a system is globally asymptotically stable, then all the solutions tend to the origin. Thus, suppose we triangulate the space so that the origin is contained in triangle '1'. Then we have

Lemma 1 If the system is globally asymptotically stable, then the matrix A associated with it as above has its sufficiently high powers of the form

$$A^k = \begin{pmatrix} (1 & 1 & \cdots & 1) \\ & & 0 & \end{pmatrix}.$$

Moreover, the converse is also true.

In order that the matrix A of the system satisfied the conditions of the lemma, we see that it must be possible to renumber the triangles so that A is strictly upper triangular and so we have a simple way of determining if a system is globally stable. Of course, we can also apply the idea to the case where the stability is not global. In that case, the basin of attraction is an invariant set and the structure theorem 2 says that the matrix of the system must be block upper triangular. From lemma 1 we see that the top left hand block must be strictly upper triangular if the system has basin of attraction corresponding to the triangles in this part of the matrix.

Consider next the application of the method to control theory. Thus, suppose we have a nonlinear control system of the form

$$\dot{x} = f(x,\ u)$$

where the (scalar) control u satisfies the hard constraint

$$|u| \le 1.$$

If the system is defined on a manifold M with a distinguished point (the 'origin') to which it is desired to control the system, then we consider as above a triangulation of M with simplex '1' containing the distinguished point. For any given simplex S_1 in the manifold, we now have a number of possible simplices into which the dynamics can go from S_1, corresponding to a choice of control. This is shown in Fig.1 below.

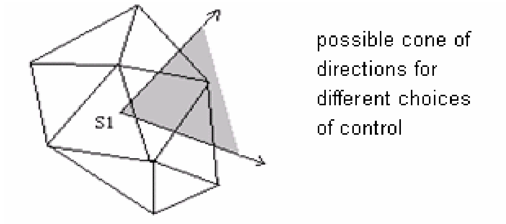

Fig.1 Possible cone of control directions

The matrix of the system now appears in the following form:

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{j1} & \cdots \\ \alpha_{21} & \cdots & \alpha_{j1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where each α is 0 or 1, depending on the choice of control. (Note that, of course, $\alpha_{jk} = 0$ if there is no choice of control which drives the simplex k to simplex j.) For each choice of control at each simplex, only one of the α 's in each column will be nonzero. Combining these ideas with lemma 1 we have the following result, which gives a simple criterion for the stabilization of nonlinear systems defined on manifolds:

Theorem 3 A system defined on a compact differentiable manifold which has a local representation in the form

$$\dot{x} = f(x,\ u)$$

and a hard constraint as above, is stabilisable if the matrix of the triangulation, for some choice of controls, is equivalent, under similarity transformation, to a strictly upper triangular matrix.

4 EXAMPLE

In this section we shall give a simple example of a system (a Van der Pol- like system in this case), together with a triangulation and a matrix representation. The system is shown (on a sphere) in Fig. 2.

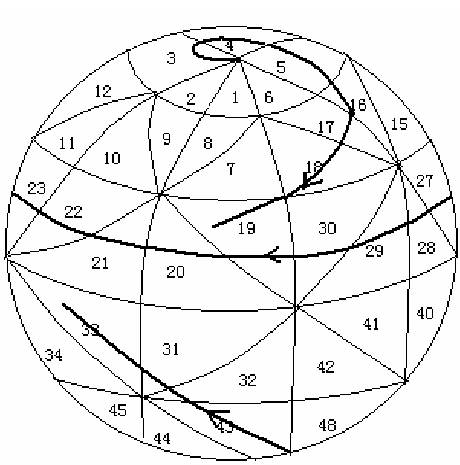

Fig. 2 A Van der Pol- like system

Rather than show the basic matrix we write down the system matrix to a high power, to illustrate the invariant cycle. In fact, it is easy to see that the system matrix to a sufficiently high power has the following form:

$$A^k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ A_1 & A_2 & A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_4 & A_5 & A_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $A_4 = A_3$, $A_5 = A_1$, $A_6 = A_2$.

5 CONCLUSION

In this paper we have shown how to represent a system in the form of a linear matrix representation using a triangulation of the manifold on which the system is defined. From this representation, it is easy to determine invariant sets and their topology. We have also shown that the approximations tend to the inverse limit of the covering defined by the triangulation. The main drawback with the method is the large size of the matrix representation, particularly for high-dimensional systems. In a future paper we shall examine more efficient representations which contain the same information.

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