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THE PASSBAND WIDTH OF THE VOLD-KALMAN ORDER TRACKING FILTER

ŠÍŘKA PROPUSTNÉHO PÁSMA VOLD-KALMANOVA ŘÁDOVÉHO FILTRU

Abstract

Even though the basic principle of the Vold-Kalman (VK) order tracking filter was published many times, the issue dealing with the problem of the filter passband setting was omitted as yet. It is known that the filtration effect of the VK-filter is achieved by solving of the linear system equations. The system matrix is sparse and the equation system solution can be based on the Cholesky factorization of the system matrix resulting in the solution of the forward reduction and backward substitution. Reduction and substitution can be considered as a filtering process. This is a main idea how to evaluate the dependence of the filter bandwidth on a parameter determining the VK-filter property.

Abstrakt

Ačkoliv základní princip Vold-Kalmanova filtru byl již několikrát publikován, problém stanovení šířky propustného pásma by opomíjen. Je známo, že filtrační efekt vyplývá z řešení soustavy lineárních rovnic. Matice soustavy rovnic je řídká a řešení příslušné soustavy může být založeno na Choleskyho faktorizaci s dopřednou redukcí a zpětnou substituci. Obě tyto operace představují filtrační proces. Toto je základní idea pro výpočet šířky propustného pásma Vold-Kalmanova filtru..

1 INTRODUCTION

Some particular class of signals consists of harmonic components that are all (or the most dominant of them) related in frequency to the fundamental frequency, e.g. engine rotational speed. These components are designated as super- or sub- harmonics (the so-called orders) of the fundamental frequency in RPM, which is measured. The paper deals with the Vold-Kalman (VK) order-tracking filter of two generations [1]. After describing the theoretical principle, main attention is focused on the control of the absolute and relative VK-filter bandwidth.

Without loss of the generality, the analysis is dealing with the tracking of just one order. Under condition that the orders are not close or crossing, the multiple orders can be tracked individually in a step-by-step way.

2 THE FIRST AND SECOND GENERATION OF THE VK-FILTER

Similarly as for the Kalman filter, which is based on the process and measurement equations, the VK-filter is based on the structural and data equations that play the similar role in the filtration effect. Both these equations are excited by the unknown functions on its right side. It is assumed that for the Kalman filter these functions are stochastic with known covariance while for the VK-filter a

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user sets only the relationship between them. The structural equation for the first and second generation of the VK-filter takes the form

First generation VK-filter

$$x(n) - 2\cos(\omega_C \, \Delta t)x(n+1) + x(n+2) = \varepsilon(n)$$

Second generation VK-filter

$$\nabla x(n) = x(n) - x(n+1) = \varepsilon(n) \qquad (1)$$

$$\nabla^2 x(n) = x(n) - 2x(n+1) + x(n+2) = \varepsilon(n)$$

$$\nabla^3 x(n) = x(n) - 3x(n+1) + 3x(n+2) - x(n+3) = \varepsilon(n)$$

where:

n – index,

$x(n)$ – amplitude and phase modulated harmonic signal as the filter output of the first generation VK filter or the envelope of the filter output for the second generation VK filter ,

Δt – sampling interval of the input data samples,

ω_C – instantaneous angular frequency of the output modulated harmonic signal,

$\varepsilon(n)$ – error term enabling a slightly change of the filter output amplitude and frequency over the time samples involved in the equation (1).

The structural equation of the second generation VK filter are shown in three possible forms differing in the order of the difference, which is equal to the error term $\varepsilon(n)$. The order of the differences plus unity gives the number of the filter poles p.

The system of the structural equations (1) containing all the samples $n = 1,...,N$ takes the following form, which can be rewritten in the matrix form

$$\mathbf{A\,x} = \boldsymbol{\varepsilon} \qquad (2)$$

Instead of observing the sampled sinusoidal signal or its envelope $x(n)$, only samples $y(n)$ are recorded. The signal $y(n)$ is combined from both the signals satisfying the structural equation (1) as well as random noise and other sinusoidal components differing in the frequency with the sinusoidal signal $x(n)$ for the first generation filter or the envelope for the second generation filter. The random noise and other sinusoidal are combined into the signal $\eta(n)$.

First generation VK-filter

$$y(n) = x(n) + \eta(n)$$

Second generation VK-filter

$$y(n) = x(n)\exp(j\Theta(n)) + \eta(n), \quad \Theta(n) = \sum_{i=0}^{n} \omega_C(i)\Delta t \qquad (3)$$

where $\Theta(n)$ is a signal phase as a result of the angular frequency integration. Formally, it can be written as N equations arranged into the matrix form

First generation VK-filter

$$\mathbf{y} - \mathbf{x} = \boldsymbol{\eta}$$

Second generation VK-filter

$$\mathbf{y} - \mathbf{C\,x} = \boldsymbol{\eta} \qquad (4)$$

$$\mathbf{C} = diag\{\exp(j\Theta(1)), \exp(j\Theta(2)), ..., \exp(j\Theta(N))\}$$

3 GLOBAL SOLUTION OF BOTH THE GENERATION VK-FILTERS

The system of the data equations (4) and the structural equations (2) is an underdetermined system for the unknown waveform $x(n)$. The additional condition for the equation solution is that the variances of the non-homogeneity terms, $\varepsilon(n)$ and the other sinusoidal components and background random noise $\eta(n)$, have to be minimal while maintaining the given relationship between them. The global solution can be found using the standard least square technique. The sum of the squares of all the unknown non-homogeneity terms for the first and second-generation algorithm can be expressed as a scalar product

$$\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = \mathbf{x}\mathbf{A}^T\mathbf{A}\,\mathbf{x} \tag{5}$$

where a row vector $\boldsymbol{\varepsilon}^T$ is a transpose of the column vector $\boldsymbol{\varepsilon}$. The sum of the squares of the signal $\eta(n)$ in both the VK-filter generations can be written in the form

First generation VK-filter

$$\boldsymbol{\eta}^T\boldsymbol{\eta} = \left(\mathbf{y}^T - \mathbf{x}^T\right)\left(\mathbf{y} - \mathbf{x}\right)$$

Second generation VK-filter

$$\boldsymbol{\eta}^H\boldsymbol{\eta} = \left(\mathbf{y}^T - \mathbf{x}^H\mathbf{C}^H\right)\left(\mathbf{y} - \mathbf{C}\mathbf{x}\right) \tag{6}$$

where the upper index H designates the complex conjugate quantities. As the matrix **C** is complex both the vectors $\mathbf{x}, \boldsymbol{\eta}$ are complex as well.

The weighted sum of the particular sums (5) and (6) gives the loss function

$$J = r^2\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} + \boldsymbol{\eta}^T\boldsymbol{\eta} \tag{7}$$

where r is a weighting factor [3, 4]. The choice of a large value for the weighting factor r leads to the highly selective filtration in the frequency domain that takes a long time to converge in amplitude. In contrast, fast convergence with low frequency resolution is achieved by choosing r small.

The first derivative of the loss function (7) with respect to the vector **x** gives a condition for the minimum of this function, which is called a normal equation.

First generation VK-filter

$$\frac{\partial J}{\partial \mathbf{x}} = 2r^2\mathbf{A}^T\mathbf{A}\,\mathbf{x} + 2\left(\mathbf{x} - \mathbf{y}\right) = \mathbf{0}$$

$$\mathbf{x} = \left(r^2\mathbf{A}^T\mathbf{A} + \mathbf{E}\right)^{-1}\mathbf{y}$$

Second generation VK-filter

$$\frac{\partial J}{\partial x} = 2r^2\mathbf{A}^T\mathbf{A}\,\mathbf{x} + 2\left(\mathbf{x} - \mathbf{C}^H\mathbf{y}\right) = \mathbf{0} \tag{8}$$

$$\mathbf{x} = \left(r^2\mathbf{A}^T\mathbf{A} + \mathbf{E}\right)^{-1}\mathbf{C}^H\mathbf{y} \tag{9}$$

The matrix equations (8) are of the same form but for an exception. The vector **y** is multiplied by $\mathbf{C}^H$, which shifts the frequency of the tracked components toward to zero. The pass band filter becomes the low pass filter. The unknown waveform in the case of the first generation VK-filter and the unknown envelope in the case of the second-generation VK-filter result from the equations (9).

The product of the matrixes, $\mathbf{A}^T$ and $\mathbf{A}$, gives a symmetric positive semidefinite matrix. The matrix $\mathbf{B} = r^2\mathbf{A}^T\mathbf{A} + \mathbf{E}$ becomes the symmetric positive definite matrix by adding the unity matrix **E**. The matrix **B** consists of the limit number of the non-zero diagonals. Therefore, it is easy to invert it. The number of the non-zero diagonals of the matrix **B** for the VK-filter of the second-generation is equal to $2p+1$, where p is the number of the filter poles. This number of the non-zero diagonals of the matrix **B** for the VK-filter of the first-generation can be designated as well by $2p+1$, where $p = 2$.

Employing the Cholesky factorization of the matrix **B** into the matrix product $\mathbf{B} = \mathbf{L}\,\mathbf{U}$, where **L** is a lower-triangular matrix and $\mathbf{U} = \mathbf{L}^T$ is an upper-triangular matrix, is the easiest way how to solve the equation system (8). The only condition for the Cholesky factorisation is that all the main minor determinants are equal to a positive value what can be easily proved. The main advantage of the Cholesky factorisation algorithm is that it saves the number of the non-zero diagonals in the triangular matrices at the value $p+1$. The solution of the system (8) is broken down into two linear equation systems, the forward reduction and backward substitution.

Forward reduction (first system)

$$z_1 = y_1/u_{1,1}\,,\; z_2 = \left(y_2 - u_{1,2}z_1\right)/u_{2,2}\,,\;\ldots$$

Backward substitution (second system)ter

$$x_N = z_1/u_{N,N}\,,\; x_{N-2} = \left(z_{N-2} - u_{N-1,N}x_N\right)/u_{N-1,N-1}\,,\;\ldots \tag{10}$$

For $j = p+1,...,N$: For $j = N-(p+1),...,1$:

$$z_j = \left(y_j - u_{j-1,j}\, z_{j-1} ... - u_{j-p,j}\, z_{j-p}\right)/u_{j,j} \qquad x_j = \left(z_j - u_{j,j+1} x_{j+1} ... - u_{j,j+p} x_{j+p}\right)/u_{j,j} \tag{11}$$

In the forward reduction, the linear equation system $\mathbf{L}\,\mathbf{z} = \mathbf{y}$ ($\mathbf{U}^T\,\mathbf{z} = \mathbf{y}$) for an unknown vector **z** is solved while in the backward substitution the unknown vector **x** of the equation system $\mathbf{U}\,\mathbf{x} = \mathbf{z}$ is evaluated.

The value of weighting factor *r* has to be limited not to lose the effect of adding unity to main matrix diagonal on the positive definiteness by rounding the diagonal elements due to the limit bit number for saving quantities in a computer memory. Taking into consideration the maximal value of the diagonal components of the matrix $r^2\mathbf{A}^T\mathbf{A}$ and 14 decimal places for double-precision computer-arithmetic, the limit value r_{MAX} for the weighting factor is shown in the table 1. This table gives also the lower limit for the relative VK-filter bandwidth, which is introduced in the next chapter.

Tab. 1 Limit value of weighting coefficient and bandwidth

Pole number:	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$\left(r^2\mathbf{A}^T\mathbf{A}\right)_{i,i}$	$2r^2$	$6r^2$	$20r^2$	$70r^2$
$r_{MAX} \approx$	7x10^6	4x10^6	2x10^6	1.1x10^6
Δf 100% >	5x10^{-6} %	0.025 %	0.5 %	2 %

4 BANDWIDTH OF THE SECOND GENERATION VK FILTER

The main idea how to evaluate the dependence of the filter bandwidth on a parameter determining the VK-filter property results from equation (11). Taking into account the reverse order of the samples $x(N),...,x(1)$ in the backward substitution, the filtration process is based on the same transfer function as for the forward reduction. Altogether the forward reduction and backward substitution results in zero-phase digital filtering analogous to the *filtfilt* function in Matlab. The roll-off of the second generation filter is equal to $-40p$ dB per decade. The dependence of the filter bandwidth on the weighting factor is evaluated for the second generation VK filter, namely for the one-pole filter version. The matrix product $\mathbf{A}^T\mathbf{A}$ is equal to

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & ... & ... & ... \end{pmatrix} \tag{12}$$

The values of the elements of the matrix $\mathbf{B} = r^2\mathbf{A}^T\mathbf{A} + \mathbf{E}$ are given by

$$b_0 = b_{j,j} = 2r^2 + 1\,,\; b_1 = b_{j,j+1} = b_{j,j-1} = -r^2 \tag{13}$$

The Cholesky factorization of the matrix **B** into the matrix product $\mathbf{B} = \mathbf{L}\,\mathbf{U}$, where **L** is a lower-triangular matrix and $\mathbf{U} = \mathbf{L}^T$ is an upper-triangular matrix, results in the formulas

$$u_{1,1} = \sqrt{b_{1,1}}\;,\text{ for } j = 2,...,N\;:\; u_{j-1,j} = b_{j-1,j}/u_{j-1,j-1}\;,\; u_{j,j} = \sqrt{b_{j,j} - u_{j-1,j}^2}\;. \tag{14}$$

The linear equation systems for the forward reduction and backward substitution are given by

$$z(1) = y(1)/u_{1,1}\;,\text{ for } j = 2,...,N\;:\; z(j) = \left(y(j) - u_{j-1,j}\,z(j-1)\right)/u_{j,j}\;. \tag{15}$$

$$x(N) = z(N)/u_{N,N}\;,\text{ for } j = N-1,...,1\;:\; x(j) = \left(z(j) - u_{j,j+1}\,x(j+1)\right)/u_{j,j}\;. \tag{16}$$

The steady-state values of the non-zero elements of the matrix **U** can be evaluated as

$$u_0 = \lim_{j\to\infty} u_{j,j} \ , \ u_1 = \lim_{j\to\infty} u_{j,j+1} \tag{17}$$

The digital filter transfer function in Z-transform resulting from (11) is defined as

$$\frac{Z(z)}{Y(z)} = \frac{1}{u_0 + u_1 z^{-1}} \tag{18}$$

It can be written for the steady-state values of the elements of the matrix **U** that are given by (14)

$$u_1 = b_1/u_0 \ , \ u_0 = \sqrt{b_0 - u_1^2} \tag{20}$$

After rearrangement it is obtained

$$u_0 u_1 = b_1 \ , \ u_0^2 + u_1^2 = b_0 \tag{21}$$

Substituting the complex quantity z by the term $\exp(j\Omega)$, where $\Omega = \omega\Delta t$ for $\Omega \in (-\pi, +\pi)$, the frequency response function based on (18) is given by

$$G\left(e^{j\Omega}\right)_{LP} = \frac{Z\left(e^{j\Omega}\right)}{Y\left(e^{j\Omega}\right)} = \frac{1}{u_0 + u_1 e^{-j\Omega}} \tag{22}$$

For the reason explained at the beginning of this chapter (two-stage filtration process), the 3 dB cut-off frequency of the one-pole filter results from the following condition

$$\left|G\left(e^{j\Omega}\right)_{LP}\right|^2 = \left|\frac{1}{u_0 + u_1 e^{-j\Omega}}\right|^2 = \frac{1}{\sqrt{2}} \ . \tag{23}$$

The relative bandwidth corresponding to the absolute cut-off frequency f_H of the low pass filter is given by the formula $\Delta f = 2 f_H / f_S$. The substitution of the exponentional term results in

$$e^{-j\Omega} = \exp(-j\pi\Delta f) = \cos(\pi\Delta f) - j\sin(\pi\Delta f) \tag{24}$$

The low pass filter frequency response function (22) is given by

$$\begin{aligned} &\frac{1}{\left|u_0 + u_1 e^{-j\Omega}\right|^2} = \frac{1}{\left|u_0 + u_1(\cos(\pi\Delta f) - j\sin(\pi\Delta f))\right|^2} = \frac{1}{u_0^2 + u_1^2 + 2u_0u_1\cos(\pi\Delta f)} = \\ &= \frac{1}{b_0 + 2b_1\cos(\pi\Delta f)} = \frac{1}{2r^2 + 1 - 2r^2\cos(\pi\Delta f)} = \frac{1}{1 + 2r^2(1-\cos(\pi\Delta f))} \end{aligned} \tag{25}$$

The last formula and equation (23) results in a formula describing the dependence of the r on Δf

$$r = \sqrt{\frac{\sqrt{2} - 1}{2(1 - \cos(\pi\Delta f))}} \tag{26}$$

5 GENERAL FORMULAS FOR EVALUATION THE FILTER BANDWIDTH

The bandwidth as the exact function of the weighting factor r for evaluation Δf is given for the first generation VK-filter by the following formula

$$\Delta f = \frac{1}{\pi}\left(\arccos\left(\cos(\Omega\Delta t) - \sqrt{\sqrt{2}-1}\Big/2r\right) - \arccos\left(\cos(\Omega\Delta t) + \sqrt{\sqrt{2}-1}\Big/2r\right)\right) \tag{27}$$

The approximation of the mentioned exact function can be performed by using substitution $\Omega = \omega_C \ \Delta t + \Delta\Omega$ followed by simplification the formula by using substitution $\cos(\Delta\Omega) \approx 1$ and $\sin(\Delta\Omega) \approx \Delta\Omega$. The approximation formula for evaluation the weighting factor r is given by

$$r \approx \frac{1}{\pi \Delta f} \frac{\sqrt{\sqrt{2}-1}}{\sqrt{1-\left(\cos\left(\omega_C \Delta t\right)\right)^2}} \tag{28}$$

For the second generation VK-filter, the weighting factor r as the exact function of the bandwidth and the approximation of this function, using the estimating formula $\cos(\pi \Delta f) \approx \left(1-(\pi \Delta f)^2/2\right)$, are summarized in the table 2.

Tab. 2 Weighting factor as a function of the bandwidth for the 2nd generation of the VK filter

Number of poles	Solution of the equation	Approximation
1	$r = \sqrt{\frac{\sqrt{2}-1}{2(1-\cos(\pi\Delta f))}}$	$r \approx \frac{0.2048624}{\Delta f}$ (29)
2	$r = \sqrt{\frac{\sqrt{2}-1}{6-8\cos(\pi\Delta f)+2\cos(2\pi\Delta f)}}$	$r \approx \frac{0.0652097315}{\Delta f^2}$ (30)
3	$r = \sqrt{\frac{\sqrt{2}-1}{20-30\cos(\pi\Delta f)+12\cos(2\pi\Delta f)-2\cos(3\pi\Delta f)}}$	$r \approx \frac{0.020756902}{\Delta f^3}$ (31)

In the case when the frequency of the tracked order is not equal to a constant value and it is not possible to evaluate the appropriate weighting factor, the loss function (7) is transferred to the form $J = \boldsymbol{\varepsilon}^T \mathbf{R}^T \mathbf{R}\, \boldsymbol{\varepsilon} + \boldsymbol{\eta}^T \boldsymbol{\eta}$, where **R** is a square diagonal matrix with diagonal elements, which are equal to the weighting factors determined for the instantaneous order frequency and the filter bandwidth, e.g. $r_{i,i} = r(f_C, \Delta f)$. The VK-filtration can work with both the absolute bandwidth Δf in Hz and the relative bandwidth $\Delta f / f_C$ in percentage to remain a constant value.

6 CONCLUSIONS

The main results of the paper are formulas describing the dependence of the weighting factor on the VK-filter bandwidth. These formulas play a key role in the software for the VK-filter and allow the setting of the filter bandwidth in either absolute value in Hz or relative value in percentage.

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